Title: Introduction to stable homotopy theory.

Abstract: In this talk, we will observe the phenomenon of stable homotopy groups of spheres and we will try to generalize it for a bigger setting by defining Spectra. We will also discuss some inter--esting properties of spectra that make them so useful.

Recall: $\pi_1(X) = [S^1 \to X]$; $\pi_i(X) = [S^i \to X]$.

(Write on board beforehand).

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0

This shows that $\pi_{i+n}(S^n)$ eventually stabilizes.

These are called i^{th} stable homotopy groups of spheres.

Is this true in general for any topological space X?

Currently we have $S^1, S^2, S^3, \ldots$, now if we start from a single topological space X, how do we get corresponding higher spaces?

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we will use smash product to define these.

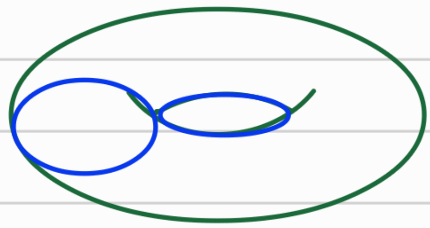
$$X \wedge Y := \frac{X \times Y}{X \vee Y}, \quad X, Y \in \mathrm{Top}_*$$

why so complicated?

Because this gives us the $\wedge \dashv \mathrm{Hom}$ adjunction.

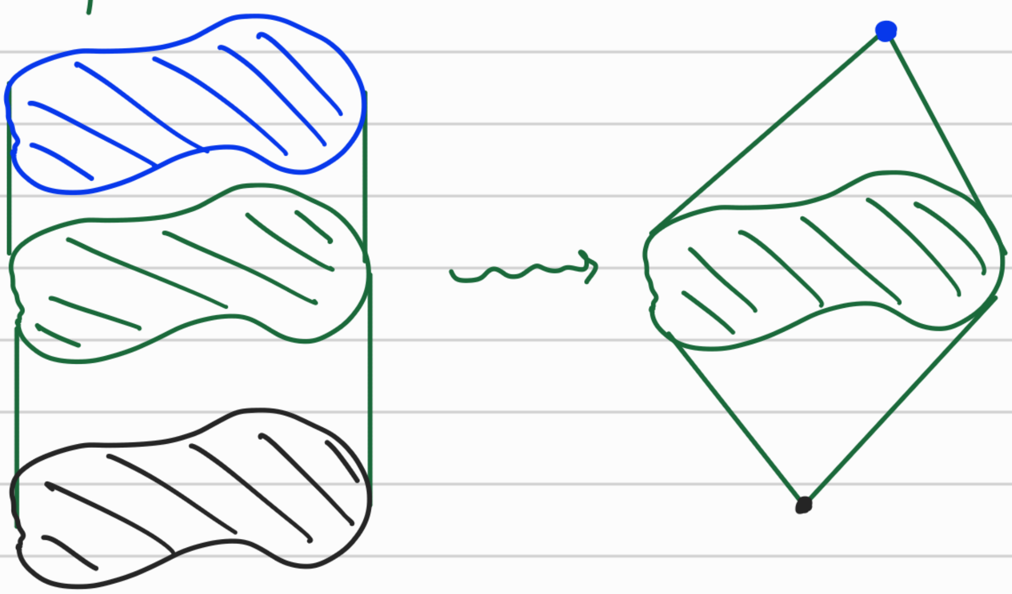
$$\mathrm{Hom}_{\mathrm{Top}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathrm{Top}_*}(X, \mathrm{Hom}_{\mathrm{Top}_*}(Y, Z))$$

E.g.: $S^1 \wedge S^1 = \dfrac{S^1 \times S^1}{S^1 \vee S^1} = S^2$



In fact, $S^n \wedge S^m = S^{n+m}$.

E.g: "Suspension": $\Sigma X := X \wedge S^1$



This gives us $\Sigma \dashv \Omega$ adjunction:

$$\mathrm{Hom}_{\mathrm{Top}_*}(\Sigma X, Z) \cong \mathrm{Hom}_{\mathrm{Top}_*}(X, \Omega Z)$$

So if we start with S^1, we would get $S^1, S^2, S^3, \dots$ by taking suspensions.

For technical reasons ← At the bottom* CGWH*

Def^n: A spectrum X is a collection $\{X_n\}_{n \in \mathbb{N}}$ of spaces with bonding maps $\Sigma X_n \xrightarrow{\partial_n} X_{n+1}$.

E.g.: 1) Sphere spectrum $\mathbb{S}$: $\partial_n = \mathrm{id}$ $\forall n$.

2) Any $X \in \mathrm{Top}_*$, $X_0 = X$, $X_1 = \Sigma X, \dots, X_n = \Sigma^n X, \dots$ & $\partial_n = \mathrm{id}$ $\forall n$. This spectrum is denoted as $\Sigma^\infty X$.

3) Given any abelian group A, we can define the Eilenberg–MacLane spectrum HA:

$$(HA)_n = K(A,n), \quad K(A,n) \underset{\downarrow}{\simeq} \Omega K(A, n+1)$$

Canonical equivalence.

$\pi_i(S^1 \to K(A,n+1))$

$$[S^i \to S^1 \to K(A,n+1)] = [S^{i+1} \to K(A,n+1)] = \begin{cases} A, & \text{if } i = n \\ 0, & \text{o.w.} \end{cases}$$

↳ from $\Sigma \dashv \Omega$

4) $U(n)$ = Group of $n \times n$ unitary matrices over $\mathbb{C}$
(Unitary: $U^H = U^{-1}$ or columns form orthonormal basis)

$$U_n \xrightarrow{in} U_{n+1} : \; in(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}; \; U = \operatorname{colim}_n U_n.$$

Then we can define the complex K-theory spectrum KU:

$$(KU)_n = \begin{cases} \mathbb{Z} \times BU & , n \text{ even} \\ U & , n \text{ odd} \end{cases}$$

Bonding maps due to Bott periodicity:

$$\Omega U \simeq \mathbb{Z} \times BU, \ \Omega(\mathbb{Z} \times BU) \simeq U.$$

In fact, any K-theory is a spectrum! This is the motivation for my study of Spectra.

Topological K-theory studies complex vector bundles over a space X, so KU is also used to study those.

5) From existing spectra, we can construct new spectra:

"Shift desuspension spectrum": $F_d \mathbb{X}$:

$$(F_d \mathbb{X})_n = \begin{cases} * & n < d \\ X_{n-d} & n \geq d \end{cases} \quad : \underbrace{\overset{0=n}{*} \to * \to \cdots \overset{d-1 = n}{*}}_{d\text{-many}} \to \overset{d=n}{X_0} \to X_1 \to \cdots$$

6) Shift operator: $d \geq 0$:

$$(Sh^d \mathbb{X})_n := \begin{cases} X_{d+n} & , d+n \geq 0 \\ * & , \text{o.w.} \end{cases} \qquad d=3: \ \underset{n=0}{X_3} \to \underset{n=1}{X_4} \to \underset{n=2}{X_5} \to \cdots$$

$d<0 \rightsquigarrow F_{|d|}$.

7) $(X \vee Y)_n := X_n \vee Y_n$ (coproduct),

$$\Sigma(X_n \vee Y_n) \xrightarrow{\cong} (\Sigma X_n) \vee (\Sigma Y_n) \xrightarrow{\partial_n^X \vee \partial_n^Y} X_{n+1} \vee Y_{n+1}.$$

8) Similarly for $(X \times Y)_n$ (product).

Def^n: Category of Spectra Sp: Obj = Spectra,

Morphism: $f: X \to Y$: $f_n : X_n \to Y_n$ s.t.

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\partial_n^X"] \arrow[d, "\Sigma f_n"'] \arrow[dr, phantom, "\circlearrowright"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\partial_n^Y"'] & Y_{n+1} \end{tikzcd} \quad \forall n.$$

Def^n: An Ω-spectrum (or fibrant spectrum) is a spectrum $\mathbb{X}$ s.t. the adjunct bonding maps: $X_n \xrightarrow{\simeq} \Omega X_{n+1}$ are weak homotopy equivalences.

* X_0 of an Ω-Spectrum is an "infinite loop space".

* E.g. 3.4.

If we have a spectrum $\mathbb{X}$, then consider:

$$\begin{tikzcd} S^m \arrow[r] \arrow[d, "\Sigma"'] & X_n \arrow[d, "\Sigma"] & \rightsquigarrow \pi_m(X_n) \\ S^{m+1} \arrow[r, dashed] \arrow[dr] & \Sigma X_n \arrow[d, "\partial_n"] & \{\Sigma \text{ is functorial}\}. \\ \pi_{m+1}(X_{n+1}). & X_{n+1} & \end{tikzcd}$$

Thm: (Freudenthal suspension thm):

$$\pi_m(X_n) \to \pi_{m+1}(X_{n+1}) \to \pi_{m+2}(X_{n+2}) \to \cdots$$

eventually stabilizes.

Def^n: i^{th} stable homotopy group of X is:

$$\pi_i^S(X) := \operatorname*{colim}_k \pi_{i+k}(X_k) \cong \pi_{i+N}(X_N),\quad N >> 0$$

{will write π_i instead of π_i^S} *Remark: $i<0$ also works.

Def^n: A map of spectra $f: X \to Y$ is a "stable equivalence" if the induced map $f_*: \pi_*^S(X) \longrightarrow \pi_*^S(Y)$ is an isomorphism (up to zig-zags).

(This is not $\underbrace{\text{levelwise}}_{\text{As in levels of } X_n}$, but levelwise $\Rightarrow$ stable equi.)

contains all objects.

Def^n: Let $W \subseteq Sp$ be the wide subcategory of stable equivalences. Then "stable homotopy category":

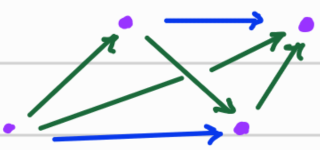
$ho\,Sp = Sp[W^{-1}]$ is the localisation of Sp at W.

{stable equi. $\longmapsto$ iso. $\in ho\,Sp$}.

{As discussed by Chris Kapulkin a couple of weeks back}.

Study of this is "stable homotopy theory".

stable equivalences satisfy 2-out-of-6 (& hence 2-out-of-3) property and can form a homotopical category.



{$\longrightarrow \in W \Rightarrow \longrightarrow \in W$}.

$ho\,Sp$ = standard notion of homotopy cat. of a homotopical category.

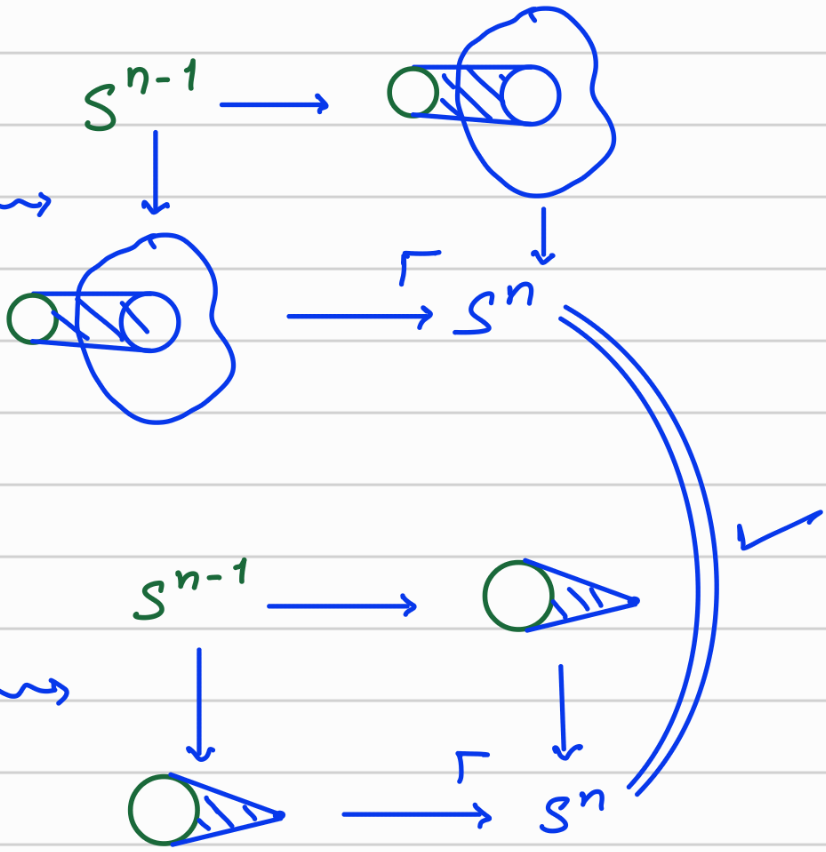
Why should one study/use Sp?

1) Homotopy pushouts = Homotopy pullbacks.

What are homotopy limits/colimits?

Motivation:
$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] \arrow[rr, bend left, "\simeq"] & D^n \arrow[d] \arrow[rr, bend left, "\simeq"] & S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\
D^n \arrow[r] \arrow[rr, bend right, "\simeq"'] & S^n \arrow[ul, phantom, "\ulcorner"] \arrow[rr, bend right, "\not\simeq"'] & * \arrow[r] & * \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\
D^n \arrow[r] & \square \arrow[ul, phantom, "h\ulcorner"]
\end{tikzcd} \rightsquigarrow$$



$\wr\!\simeq$

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \square \arrow[ul, phantom, "h\ulcorner"]
\end{tikzcd} \rightsquigarrow$$

Def^n: A left deformation of $\mathcal{C}$ is an endofunctor $Q: \mathcal{C} \to \mathcal{C}$ together with a natural w.e. $q: Q \overset{\sim}{\Rightarrow} id_{\mathcal{C}}$.

A left deformation of a functor $F: \mathcal{C} \to \mathcal{D}$ is a left defo. $q: Q \Rightarrow id_{\mathcal{C}}$ of $\mathcal{C}$ and a choice of $\mathcal{C}_Q$ s.t.

F is homotopical on $\mathcal{C}_Q$ {$\mathcal{C}_Q$ is a full subcat. of $\mathcal{C}$ that contains image of Q}.

Thm: If $q : Q \Rightarrow id$ is a left defo. of $F : \mathcal{C} \to \mathcal{D}$, then

$\mathbb{L}F = F \circ Q$ is a left derived functor of F.

Def^n: The homotopy colimit (if it exists) is a left derived functor

$$\mathrm{hocolim} := \mathbb{L}(\mathrm{colim} : \mathcal{C}^J \to \mathcal{C}).$$

{To produce it, find a left defo.: $Q : \mathcal{C}^J \to \mathcal{C}^J$, $q : Q \Rightarrow id$ of colim}.

2) So in particular, cofiber sequences = fiber sequences. "Stable".

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ * \arrow[r, "h\ulcorner"] & \mathrm{cof}(f) \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathrm{fib}(g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner h"] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

Usefulness $\rightsquigarrow \cdots \to \pi_{n+1}(\mathrm{cof}(f)) \to \pi_n X \to \pi_n Y \to \pi_n(\mathrm{cof}(f)) \to \cdots$ LES.

3) Mayer-Vietoris-type LES:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r, "h\ulcorner"] & B \end{tikzcd} \rightsquigarrow \cdots \to \pi_{n+1} B \to \pi_n X \to \pi_n A \oplus \pi_n Y \to \pi_n B \to \cdots$$

4) Hurewicz thm: (In degree ≥ 2): If a topological space

X is $(n-1)$-connected (i.e., $\pi_i(X) = 0 \ \forall i \leq n-1$) for $n \geq 2$, then

$$\pi_n X \xrightarrow{\cong} H_n(X) \text{ via Hurewicz homomorphism.}$$

i.e., for $n \geq 2$, the first non-trivial homotopy group occurs at the same level where first non-trivial homology group occurs, and they are isomorphic.

Hurewicz homomorphism:

$$\Phi : \pi_k(X, x) \longrightarrow H_k(X)$$

$$(f : S^k \to X) \longmapsto f_*[S_k]$$

where $[S_k] \in H_k(S^k) \cong \mathbb{Z}$ is the "fundamental class", i.e., generator of this $\mathbb{Z}$.

Do we have a similar thm for spectra?
Ans: Yes! But what is $H_n(X)$?

Like $\pi_n(X)$, we can have the following chain of maps induced from the bonding maps $\Sigma X_n \to X_{n+1}$:

$$H_{k+n}(X_n; G) \longrightarrow H_{k+n+1}(X_{n+1}; G) \longrightarrow H_{k+n+2}(X_{n+2}; G) \longrightarrow \cdots$$

and again, $H_k(X; G) := \operatorname{colim}_n H_{k+n}(X_n; G)$.

Thm: $H_k(X; G) = \pi_k(\underbrace{X \wedge HG}_{\text{Complicated}})$.

Examples: i) $H_k(\Sigma^\infty X; \mathbb{Z}) = H_k(X; \mathbb{Z}) = H_k(X)$ as Σ^∞ just shifts ordinary homology.

ii) $H_k(\$, \mathbb{Z}) = H_k(S^0) = \begin{cases} \mathbb{Z} & , k=0 \\ 0 & , o.w. \end{cases}$

With this definition of homology of a spectrum, we have the Hurewicz thm for spectrum.

In this case, we get the Hurewicz homomorphism by considering Hurewicz homomorphism for spaces level-wise, and then taking colimit.

Corollary: If X, Y are bounded below, then $f: X \to Y$ is a stable equivalence iff it induces isomorphisms on homology groups.

5) $Ho(Sp)$ is an additive category:

1) admits finite products & coproducts. (E.g. 7,8).
2) has a zero object. (point spectrum).
3) The unique map $X \amalg Y \xrightarrow{\psi_{X,Y}} X \times Y$ is iso. $\forall X, Y \in Ob$.
4) $Hom(X,Y) \times Hom(X,Y) \to Hom(X,Y)$

$(f,g) \longmapsto X \xrightarrow{\Delta} X \times X \xrightarrow{(f,g)} Y \times Y \xrightarrow{\psi^{-1}_{Y,Y}} Y \amalg Y \xrightarrow{\nabla} Y$

defines a group law with $X \to * \to Y$ as id.

3') Stable equivalence $X \vee Y \to X \times Y \rightsquigarrow$ iso in $Ho(Sp)$.
4') Existence of inverse.

6) $Ho(\mathcal{S}p)$ is a triangulated category.

1) Additive category. ✓
2) Translation functor: $X \overset{\sim}{\longmapsto} X[1]$ $\quad X \longmapsto \Sigma X$
3) Distinguished triangles: $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ s.t.
fiber/cofiber sequences.

TR1) • Any morphism $X \xrightarrow{f} Y$ can be extended to a D.Δ.
• D.Δ are stable under iso. cof(f) is homotopical.
• $X = X \longrightarrow * \longrightarrow X[1]$ is a D.Δ $\forall X \in ob$.

TR2) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a D.Δ $\Leftrightarrow$ $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is so.
Puppe sequence.

TR3)
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

Universal prop. of cof(·)
also by naturality
of Puppe sequence.

TR4) (Octahedral axiom).
$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Z \arrow[rr] \arrow[dr] & & Z/Y \arrow[rr] \arrow[dr] & & Y/X[1] \\
& Y \arrow[ur] \arrow[dr] & & Z/X \arrow[ur] \arrow[dr] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[ur] \arrow[rr] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

Universal prop. & pasting law.

For more information:
Reference: Spectra and stable homotopy theory.
- Cary Malkiewich.

CGWH = Compactly generated Weakly Hausdorff spaces.

CG: If $C \subseteq X$ is closed iff $f^{-1}(C) \subseteq K$ is closed for every cpt Hausdorff space K & cts map $f: K \to X$.

WH: For such f, $f(K)$ is closed.

Almost all the spaces that we usually encounter are CGWH.

Why not just Top? Because (among other possible reasons, we need the category to be closed under products. For e.g. product of CW-cpxes is not a CW-cpx under product topo. but it is a CW-cpx if we take CGWH closer & consider a finer topology).