Tittle: Homotopy theory of Model categories.

Abstract: The talk is based on a paper by W.G. Dwyer & J. Spalinski. In this talk, I define Model categories, see some examples, then I define homotopy category of Model categories in a couple of ways and see some applications of this homotopy theory.

Motivation: Study of Ch_R = Homological algebra, in other words, homotopy theory associated to Ch_R, i.e., homotopy category $Ho(Ch_R)$ is "Homological algebra". We generalise this notion to a bigger class of categories: "Model categories" and we generalise the notion of homological algebra to homotopical algebra.

Because we can't do homological algebra on unbounded chain complexes, so we use the model structure which is consistent with bounded Ch_R.
(some details at the end).

Def^n (Model Category)

A Model category is a category $\mathcal{C}$ with three classes of maps:

i) Weak equivalences ($\xrightarrow{\sim}$)
ii) fibrations ($\twoheadrightarrow$)
iii) Cofibrations ($\hookrightarrow$)

each of which is closed under composition & contain all identity maps.

A map which is both a fibration (resp. cofibration) & a w.e. is called an acyclic fibration (resp. acyclic cofibration).

We require the following axioms:

MC1) Finite limits & colimits exist

MC2) If two out of three maps $f, g, g\circ f$ are w.e., then so is the third (given that $g\circ f$ is defined).

MC3) If f is a retract of g & g is a fibration, cofibration or a w.e. then so is f.

Def^n (Retract): An object X of a category $\mathcal{C}$ is said to be a retract of an object Y if $\exists$ morphisms

$i: X \to Y$ & $r: Y \to X$ such that $r \circ i = id_X$

E.g. In Mod_R, X is a retract of Y if $Y \cong X \oplus Z$ for an R-mod Z.

Morphism f is a retract of g in $\mathcal{C}$ if f is a retract of g as an object of $Mor(\mathcal{C})$ {Category of morphisms of $\mathcal{C}$}; i.e.,

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"'] & Y' \arrow[r, "r'"'] & X' \end{tikzcd}$$

commutes such that $r \circ i = id_X$; $r' \circ i' = id_{X'}$.

MC4) Given a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

a lift h exists in either of the following two situations:

i) i is a cofibration & p is an acyclic fibration.
ii) i is an acyclic cofibration & p is fibration.

MC5) Any map f can be factored in two ways:

i) $f = p \circ i$ where i is a cofibra^n & p is an acyclic fibra^n.
ii) $f = p \circ i$ where i is an acyclic cofibra^n & p is a fibra^n.

MC1) $\Rightarrow$ $\mathcal{C}$ has both initial (ϕ) and terminal ($*$) object.

So, $A \in Ob(\mathcal{C})$ is said to be cofibrant if $\phi \to A$ is a cofibration, and fibrant if $A \to *$ is a fibration.

Later: $Hom_{Ho(\mathcal{C})}(A,B) = Hom_{\mathcal{C}}(A,B)/_{\sim}$ only if A is cofibrant & B is fibrant.

Examples: 1) Ch_R: Non-negative chain complexes of R-mod.

A map $f: M \to N$ is defined to be

i) a w.e. if f induces isomorphisms on homology grps.
ii) a fibration if for each $k \geq 1$, the map $f_k: M_k \to N_k$ is an epimorphism.
iii) a cofibration if for each $k \geq 0$, the map $f_k: M_k \to N_k$ is a monomorphism with a projective R-mod as its cokernel.

Cofibrant objects: M s.t. each M_k is a projective R-mod.

2) Let $\mathcal{C}$ be a Model category, then the opposite category $\mathcal{C}^{op}$ with $f^{op}: X \to Y$ can be a

i) w.e. if $f: X \to Y$ is a w.e. in $\mathcal{C}$.
ii) cofibration if $f: X \to Y$ is a fibration in $\mathcal{C}$.
iii) fibration if $f: X \to Y$ is a cofibration in $\mathcal{C}$.

3) Let $\mathcal{C}$ be a Model category, $A \in Ob(\mathcal{C})$, $A \downarrow \mathcal{C}$: $Ob = f: A \to X$ in $\mathcal{C}$, then $h: (A \to X) \to (A \to Y)$ in $A \downarrow \mathcal{C}$ can be a

i) w.e. if $\tilde{h}: X \to Y$ is a w.e. in $\mathcal{C}$.
ii) cofibration if $\tilde{h}: X \to Y$ is a cofibra^n in $\mathcal{C}$.
iii) fibration if $\tilde{h}: X \to Y$ is a fibration in $\mathcal{C}$.

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[d, phantom, "\xrightarrow{h}"] & \\ X \arrow[rr, dashed, "\tilde{h}"'] & & Y \end{tikzcd}$$

Similarly $\mathcal{C} \downarrow A$: This gives rise to "fiberwise homotopy theory".

Def^n (LLP/RLP): A map $i: A \to B$ is said to have the left lifting property (LLP) w.r.t. $p: X \to Y$ (& p is said to have RLP w.r.t. i) if a lift exists in any (i.e., any f, g) diag of the form:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

Proposition: Let $\mathcal{C}$ be a Model category.

i) The cofibrations in $\mathcal{C}$ are the maps which have LLP w.r.t. acyclic fibrations.

ii) The acyclic cofibrations in $\mathcal{C}$ are the maps which have LLP w.r.t. fibrations & OPP.

Pf: Using MC4, MC5, MC3.

Remark: So to define a Model category, it is enough to define w.e., one of fibrations & cofibrations.

Model category structure on Top: → Model structure on unbdd Ch_R.

A map $f: X \to Y$ in Top is called a

i) w. (homotopy) e. if for each basepoint $p \in X$, the map $f_*: \pi_n(X, x) \to \pi_n(Y, f(x))$
$\alpha: S^n \to X \mapsto f\circ\alpha: S^n \to X \to Y$ is an iso. of grps for $n \geq 1$ & a bijection of pointed sets for $n = 0$.

ii) Serre fibration if for each CW-cpx A, the map f has the RLP w.r.t. the inclusion $A \times \{0\} \to A \times [0,1]$:

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

iii) Cofibration if it has a LLP w.r.t. acyclic fibrations.

($\Leftrightarrow$ f is a retract of a map $X \to Y'$ where Y' is obtained from X by attaching cells).

Remark: Every topological space is w.e. to a CW-cpx. i.e., Ho(Top) and Ho(CW-cpx) are equivalent.

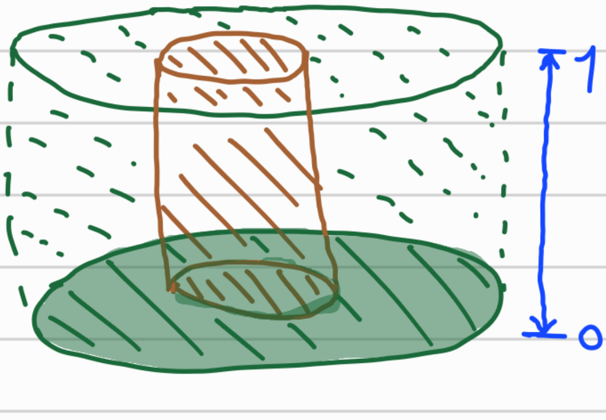
Another model category structure on Top:

A map $f: X \to Y$ in Top is called a

i) w.e. if f is a homotopy equivalence.
{$g, h: M \to N$ are homotopy equivalent if $\exists$ a cts map $H: M \times [0,1] \to N$ s.t. $H(x,0) = g(x)$, $H(x,1) = h(x)$. "$g \overset{H}{\simeq} h$". $f: X \to Y$ is a homotopy equivalence if $\exists\ \tilde{f}: Y \to X$ s.t. $H_X: X \times [0,1] \to Y$, $H_Y: Y \times [0,1] \to X$ s.t. $f \circ \tilde{f} \overset{H_Y}{\sim} Id_Y$ & $\tilde{f} \circ f \overset{H_X}{\sim} Id_X$.

ii) cofibration if f is a closed Hurewicz cofibration, i.e., a subspace inclusion $i: A \to B$, s.t. A is closed subspace of B and i has the homotopy extension property, i.e., for every space Y, a lift exists in the following diag:

$$\begin{tikzcd} B \times \{0\} \cup A \times [0,1] \arrow[r] \arrow[d] & Y \arrow[d] \\ B \times [0,1] \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$



iii) fibration is a Hurewicz fibration, i.e., a map $f: X \to Y$ s.t. f has a homotopy lifting property, i.e., for every space A, $\exists$ a lift in the following diag:

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ A\times[0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

These two model structures are different.
E.g.: Warsaw circle (W): Cpt subspace of $\mathbb{R}^2$ given by $\{y=\sin(1/x) \mid 0<x\leq 1\} \cup \{(0,y) \mid y\in[-1,1]\} \cup \{$an arc joining $(0,-1),(1,\sin(1)\}$. Then the map $W\to *$ is a w. (homotopy) e. but not a homotopy equivalence, i.e., it is a w.e. as first structure but not as second.

Homotopy category $Ho(\mathcal{C})$ of a Model category $\mathcal{C}$.

Def^n (Cylinder object). A cylinder object for A is an object $A\wedge I$ of $\mathcal{C}$ together with a diag

$$A \amalg A \xrightarrow{i} A\wedge I \xrightarrow{\sim} A$$

$A\amalg A$ is a coproduct, which exists due to MC1;

$$\begin{tikzcd} A \arrow[dr, "in_0"] \arrow[ddr, bend right, "id_A"'] & & A \arrow[dl, "in_1"'] \arrow[ddl, bend left, "id_A"] \\ & A\amalg A \arrow[d, dashed, "id_A+id_A"] & \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A\amalg A \arrow[rr] \arrow[dr, dashed, "i"'] & & A \\ & A\wedge I \arrow[ur, dashed, "\sim"'] \arrow[u, phantom, "MC5"] & \end{tikzcd}$$

Universal prop. of $A\amalg A$

Again, due to universal prop. of $A\amalg A$: $i = \underbrace{i_0}_{i\circ in_0} + \underbrace{i_1}_{i\circ in_1}$.

Def^n: Two maps $f,g : A \to X$ in $\mathcal{C}$ are said to be left homotopic ($f \overset{l}{\sim} g$) if $\exists$ a cylinder object $A \wedge I$ for A s.t. the sum map $f+g : A \amalg A \to X$ extends to a map $H : A \wedge I \to X$, i.e., $\exists$ a map $H : A \wedge I \to X$ s.t. $H(i_0 + i_1) = f+g$
H is "left homotopy between f and g" (via $A \wedge I$).

$$\begin{tikzcd} A \amalg A \arrow[r, "f+g"] \arrow[d, "i"'] & X \arrow[d] \\ A \wedge I \arrow[r] \arrow[ur, dashed, "H"] & * \end{tikzcd}$$

Being left homotopic is an equivalence relation (reflexive, symmetric, transitive).

For right homotopy, we use something called a path object X^I instead of a cylinder object.

A path object for $X \in Ob(\mathcal{C})$ is an object X^I together with a diagram: $X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$.

$$\begin{tikzcd} & X \arrow[d, dashed, "(id_X, id_X)"] \arrow[ddl, bend right, "id_X"'] \arrow[ddr, bend left, "id_X"] & \\ & X \times X \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & X \end{tikzcd} \longrightarrow \begin{tikzcd} X \arrow[rr] \arrow[dr, "\sim"'] & \text{MC5} & X \times X \\ & X^I \arrow[ur, "p"'] & \end{tikzcd}$$

Def^n: Two maps $f,g : A \to X$ are said to be right homotopic ($f \overset{r}{\sim} g$) if $\exists$ a path object X^I for X s.t. the product map $(f,g) : A \to X \times X$ lifts to a map $H : A \longrightarrow X^I$.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X^I \arrow[d, "p"] \\
A \arrow[r, "(f,g)"'] \arrow[ur, dashed, "H"] & X \times X
\end{tikzcd}$$

Remark: $\mathcal{C} = Top$, with the first model category structure define above (maybe also the second)
$\Rightarrow A \wedge I = A \times [0,1]$ (as expected);
$X^I = Hom_{Top}([0,1], X)$: cts maps : $[0,1] \to X$.

Again, $\overset{r}{\sim}$ is an equivalence relation.

Lemma: Let $f, g : A \to X$ be maps.
(i) If A is cofibrant, then $f \overset{l}{\sim} g \Rightarrow f \overset{r}{\sim} g$.
(ii) If X is fibrant, then $f \overset{r}{\sim} g \Rightarrow f \overset{l}{\sim} g$.

If A is cofibrant & X is fibrant, then denote $\overset{l}{\sim}, \overset{r}{\sim}$ by $\sim$: an equivalence relation on $Hom_{\mathcal{C}}(A,X)$.

$$\pi(A,X) := Hom_{\mathcal{C}}(A,X)/\sim .$$

For any $X \in Ob(\mathcal{C})$, we have $\phi \to X$, i.e., we obtain
$$\begin{tikzcd}
\phi \arrow[r] \arrow[d, hook, "MC5"'] & X \\
QX \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$
a cofibrant object QX which is w.e. to X.

Similarly for fibrant:
$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "\sim"'] & * \\
RX \arrow[ur, two heads, "MC5"'] &
\end{tikzcd}$$
$RX \rightsquigarrow$ fibrant.

Given a map $f: X \to Y$ in $\mathcal{C}$, $\exists\ \tilde{f}: QX \to QY$ s.t.

$$\begin{tikzcd} QX \arrow[r, "\tilde{f}"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & QY \arrow[d, "\sim"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Pf: Apply MC4 to
$$\begin{tikzcd} \phi \arrow[r, hook] \arrow[d, hook] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "f \circ p_X"'] \arrow[ur, dashed, "\exists \tilde{f}"] & Y \end{tikzcd}$$

Similarly:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sim"', "i_X"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "i_Y"', "\sim"] \\ RX \arrow[r, "\bar{f}"'] & RY \end{tikzcd}$$

Pf:
$$\begin{tikzcd} X \arrow[r, "i_Y \circ f"] \arrow[d, "\sim"', "i_X"] & RY \arrow[d] \\ RX \arrow[r, two heads] \arrow[ur, dashed, "\exists \bar{f}"'] & * \end{tikzcd}$$

Def^n: The homotopy category $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with

$Ob(Ho(\mathcal{C})) = Ob(\mathcal{C})$;

$Hom_{Ho(\mathcal{C})}(X,Y) = \pi(RQX, RQY)$

Another definition as a universal property:

Let $W \subseteq Mor(\mathcal{C})$, a localisation of $\mathcal{C}$ w.r.t. W is a functor $F: \mathcal{C} \to D$ s.t. $F(f)$ is an iso for each $f \in W$, and F is initial in the category of functors with this property, i.e.,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "F'"'] & D \arrow[d, dashed, "\exists! \tilde{F}'"] \arrow[dl, phantom, "\circlearrowleft"] \\ & D' \end{tikzcd}$$

When W = Class of w.e.: $\mathcal{C} \xrightarrow{\gamma} Ho(\mathcal{C})$ is universal.
i.e., For $f \in Mor(\mathcal{C})$, $\gamma(f)$ is an iso $\Leftrightarrow$ f is a w.e.

Applications and more examples:

1) Simplicial sets: Δ be a category:
$Ob(\Delta)$ = Ordered sets $[n] = \{0,1,2,\dots,n\}$.
$Mor(\Delta)$ = Order preserving maps.

Category sSet of simplicial sets:
$Ob(sSet)$ = Functors $\Delta^{op} \to Set$, i.e., contravariant functors: $\Delta \to Set$
$Mor(sSet)$ = Natural transformations.

Let $X \in sSet$, denote $X([n])$ by X_n: "set of n-simplices of X".
Let Δ_n denote the standard topological n-simplex (the space of formal convex linear combinations of points in $[n]$).

Let $Y \in Top$, define $Sing(Y) \in sSet$ as follows:
$Sing(Y)_n := \{$cts maps from $\Delta_n \to Y\}$.

$Sing: Top \to sSet$ has a left-adjoint: $X \mapsto |X|$ "geometric realisation of X".

Call a map $f: X \to Y$ of sSet a

i) w.e. if $|f|$ is a w. (homotopy) e. of Top.
ii) cofibration if each map $f_n: X_n \to Y_n$ is a monomorph.
iii) fibration if f has the RLP w.r.t. acyclic cofibrations.

Thm (9.7): Let $\mathcal{C}$ and $\mathcal{D}$ be model categories &
$F: \mathcal{C} \Leftrightarrow \mathcal{D}: G$ {i.e., F is a left-adjoint of G & opp}

i.e., $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \overset{\text{set}}{\cong} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$.

s.t. i) F preserves cofibrations & G preserves fibrations

ii) For each cofibrant obj. A of $\mathcal{C}$, fibrant obj X of $\mathcal{D}$, a map $f: A \to G(X)$ is a w.e. iff its adjoint $\tilde{f}: F(A) \to X$ is a w.e. in $\mathcal{D}$

then $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ are equivalent categories.

$|\cdot|$: sSet $\Leftrightarrow$ Top : Sing satisfy both conditions of the thm.

This shows category of sSet is a good category of algebraic or combinatorial models for study of ordinary homotopy theory.

This can be extended to other categories in an obvious way: $\mathcal{C}$: Category; $s\mathcal{C}$: $\mathrm{Ob}(s\mathcal{C}) = \{\Delta^{op} \to \mathcal{C}\}$

E.g: $\mathcal{C} = \mathrm{Grp} \Rightarrow s\mathcal{C} =$ Simplical grps.

Suppose $\mathcal{C}$ has an underlying set, i.e., we have a forgetful functor $U: \mathcal{C} \to \mathrm{Set}$, call a map $f: X \to Y$ in $s\mathcal{C}$ a

i) w.e. if $U(f)$ is a w.e. in sSet.

ii) fibration if $U(f)$ is a fibration in sSet.

iii) cofibration if f has the LLP w.r.t. acyclic cofibrations

E.g: $\mathcal{C} =$ Grp, Ab, associative algebras, Lie algebras, commutative algebras.

E.g. $\mathcal{C} =$ Commutative rings: "André-Quillen cohomology".

Highlights of homological algebra:

* Left-exact functors: $0\to A\to B\to C\to 0$ exact $\Rightarrow$
$0\to FA\to FB\to FC$ exact or $FA\leftarrow FB\leftarrow FC\leftarrow 0$ exact & opp.
E.g: $\mathrm{Hom}(D,-)$ & $\mathrm{Hom}(-,D)$.
Right-exact functors: opp: E.g: $-\otimes D$; $D\otimes -$.

* $\exists$ always a projective module $P_0 \twoheadrightarrow M \to 0$. $R^{\oplus M}$ ✓

* $\exists$ always a projective (simmi. injective) resolution
$P_3\to P_2\to P_1\to P_0\to M\to 0$ (i.e. an exact seq with P_i's being proj. R-mod)

* Right-derived functor: Let F be left-exact, covariant/contraviant, then consider $M\to E_\bullet$ (inje. res) / $P_\bullet\to M$ (proj. res), Applying F, we get $0\to FM\to FE_0\to FE_1\to FE_2\to\cdots$ /
$0\to \cancel{FM}\to FP_0\to FP_1\to FP_2\to\cdots$
$$R^iF(M):=H^i(FE_\bullet)=H^i(FP_\bullet)$$
$H^0=0$ $\because$ F is left-exact; but not necessarily for $i>0$.
Simmi. for left-derived functor: left-exact $\rightsquigarrow$ right-exact; cohomology $\rightsquigarrow$ homology.

doesn't matter

* $\mathrm{Ext}^i(D,M):=R^iF(M)$ when $F=\mathrm{Hom}(-,D)$ or $\mathrm{Hom}(D,-)$
* $\mathrm{Tor}^i(M,D):=L^iF(M)$ when $F=-\otimes D$ or $D\otimes -$.

* So we need boundedness to construct the resolutions.