# K-theory of a Waldhausen category


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# Introduction

Gadget

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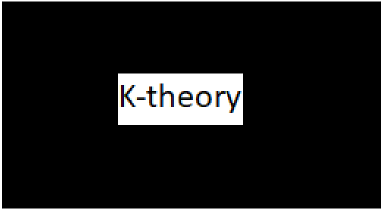


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Space with interesting homotopy groups

# History

| | | | |
|---|---|---|---|
| Low dimensional algebraic K-theory | Rings | $\mathrm{K}_0, K_1, K_2$ | 1957-67 |
| Higher algebraic K-theory | Rings | Quillen's +-construction | 1971 |
| K-theory of Schemes | Small Exact Categories | Quillen's Q-construction | 1972 |
| K-theory of Spaces | Waldhausen Categories | Waldhausen's $\mathrm{S}_\bullet$-construction | 1978 |

# Waldhausen categories

## Definition 1

Let $\mathcal{C}$ be a category equipped with a subcategory $co = co(\mathcal{C})$ of morphisms in the category $\mathcal{C}$ called cofibrations ($\rightarrowtail$). The pair $(\mathcal{C}, co)$ is called a category with cofibrations if the following axioms are satisfied:

## Definition 1

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1. Every isomorphism in $\mathcal{C}$ is a cofibration.
2. There is a zero object, $0$ in $\mathcal{C}$, and the unique morphism $0 \rightarrowtail A$ in $\mathcal{C}$ is a cofibration for every $A \in Ob(\mathcal{C})$. (i.e., every object of $\mathcal{C}$ is cofibrant).
3. If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \bigcup_A C$ of these two maps exists in $\mathcal{C}$ and $C \rightarrowtail B \bigcup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \bigcup_A C \end{array}$$

## Remarks

1. Coproduct $B \coprod C$ of any two objects $B, C \in Ob(\mathcal{C})$ exists.
   Since, $B \coprod C = B \bigcup_0 C$.
2. Every cofibration $A \rightarrowtail B$ in $\mathcal{C}$ has a cokernel $B/A$.
   Since, $B/A = B \bigcup_A 0$.
3. We refer to $A \rightarrowtail B \twoheadrightarrow B/A$ as a cofibration sequence in $\mathcal{C}$.

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## Example 2

1. The category $\mathbf{R}-\mathbf{Mod}$, for any ring $R$ is a category with cofibrations:
   The cofibrations are the injective maps.
2. In fact, any exact category, hence any abelian category is naturally a category with cofibrations:
   The cofibrations are the monomorphisms.

## Definition 3

A Waldhausen category $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called weak equivalences (abbreviated w.e. and indicated with $\xrightarrow{\sim}$) satisfying the following axioms:

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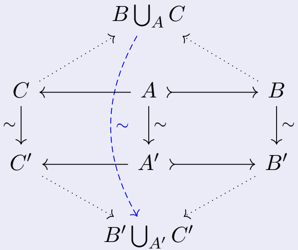
1. Every isomorphism in $\mathcal{C}$ is a w.e.
2. Weak equivalences are closed under composition.
   (So we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$.)
3. Gluing axiom:




   The induced map $B \bigcup_A C \to B' \bigcup_{A'} C'$ is also a weak equivalence.

## Definition 3

- Extension axiom:

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow{\scriptstyle\sim} & & \color{blue}{\downarrow{\scriptstyle\sim}} & & \downarrow{\scriptstyle\sim} \\ A' & \rightarrowtail & B' & \longrightarrow & B'/A' \end{array}$$

If $A \to A'$ and $B/A \to B'/A'$ are w.e. then so is $B \to B'$.

### Definition 4

A Waldhausen category $\mathcal{C}$ is called saturated if $A \xrightarrow{f} B \xrightarrow{g} C$, $g \circ f$ is a w.e., then $f$ is a w.e. if and only if $g$ is.

### Remark

We will consider only saturated Waldhausen categories, and hence we will just call them Waldhausen categories by abuse of language.

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## Example 5

The category of bounded above $(k \geq 0)$ chain complexes over a ring $R$, $\mathbf{Ch}_R$ is a Waldhausen category by defining a map $f : M \to N \in Hom_{Ch_R}(M, N)$ is

- a w.e. if $f$ induces isomorphism on homology groups.
- a cofibration if for each $k \geq 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective module as its cokernel.

### Example 6

Given a space $X$, consider the category $\mathcal{R}(X)$ of spaces that retract to $X$.

- Cofibrations are Serre cofibrations, as in the model structure.
- W.E. are maps that induce isomorphisms for some chosen homology theory.

### Example 7

Any category with cofibrations $(\mathcal{C}, co)$ may be considered as a Waldhausen category in which the category of weak equivalences is the category $\text{iso}(\mathcal{C})$ of all isomorphisms.

# $S_\bullet$-construction

- We will now see $S_\bullet$-construction. $S$ stands for Segal as in Graeme B. Segal. Segal gave a similar construction for additive categories but it was reinvented by Waldhausen for Waldhausen categories.

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- For any category $\mathcal{C}$, the arrow category $Ar\mathcal{C}$ is the category with $Ob(Ar\mathcal{C}) =$ Morphisms in $\mathcal{C}$, a morphism from $f : a \to b$ to $g : c \to d$ is a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} a & \longrightarrow & c \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ b & \longrightarrow & d \end{array}$$

- Consider $[n] = \{0 \leftarrow 1 \leftarrow \cdots \leftarrow n\}$ as a category, and the arrow category $Ar([n]^{op})$.

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- Consider $[n] = \{0 \leftarrow 1 \leftarrow \cdots \leftarrow n\}$ as a category, and the arrow category $Ar([n]^{op})$.
- For e.g. in $Ar([11]^{op})$ there is a unique morphism from the object $(2 \to 4)$ to $(3 \to 7)$ and no morphism in the other way.

## Definition 8

Let $\mathcal{C}$ be a category with cofibrations. Then $S\mathcal{C} = \{[n] \mapsto S_n\mathcal{C}\}$ is the simplicial category which in degree $n$ is the category $S_n\mathcal{C}$ of functors $C : Ar([n]^{op}) \to \mathcal{C}$ satisfying the following properties: To appendix

1. For all $j \geq 0$, $C(j = j) = 0$.
2. If $i \leq j \leq k$, then $C(i \leq j) \rightarrowtail C(i \leq k)$ is a cofibration, and

$$\begin{array}{ccc} C(j = j) & \longrightarrow & C(j \leq k) \\ \uparrow & & \twoheaduparrow \\ C(i \leq j) & \rightarrowtail & C(i \leq k) \end{array}$$

   is a pushout.

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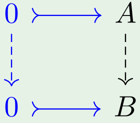
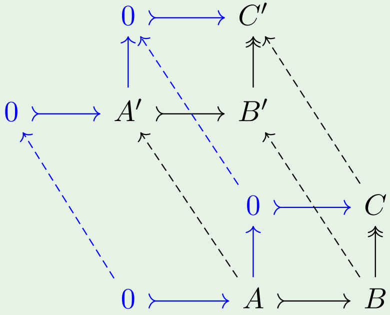



- $S_2\mathcal{C}$:



## Example 9

- $S_3\mathcal{C}$:

$$
\begin{array}{ccccccc}
 & & & & 0 & \rightarrowtail & F \\
 & & & & \uparrow & & \twoheaduparrow \\
 & & 0 & \rightarrowtail & D & \rightarrowtail & E \\
 & & \uparrow & & \twoheaduparrow & & \twoheaduparrow \\
0 & \rightarrowtail & A & \rightarrowtail & B & \rightarrowtail & C
\end{array}
$$

# $S_2\mathcal{C}$ as a category with cofibrations

- Given a category with cofibrations $\mathcal{C}$, we can define a category called $S_2\mathcal{C}$ which has $(Ob(S_2\mathcal{C}))$ = collection of cofibration sequences, morphisms between two objects as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

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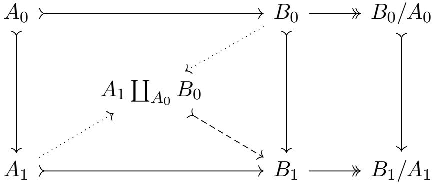
- We can define cofibrations in the category $S_2\mathcal{C}$. A map like the one above is a cofibration if the vertical maps are cofibrations and the map from $A_1 \coprod_{A_0} B_0 \to B_1$ is a cofibration.



## Remark

It can be seen that, with a similar pattern $S_n\mathcal{C}$ is a category with cofibrations for every $n \in \mathbb{N}$.
Hence, one can consider $S_\bullet(S_\bullet\mathcal{C})$ and keep on doing this. This will give us a spectrum.

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Hence, one can consider $S_\bullet(S_\bullet\mathcal{C})$ and keep on doing this. This will give us a spectrum.
However, we are not working with this spectrum in this talk. We are just considering the first level of this spectrum, i.e., we are not considering the cofibration structure over $S_n\mathcal{C}$ for $n \geq 2$.

# K-theory of Waldhausen categories

## Definition 10

Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$ is the abelian group presented as having one generator $[C]$ for each $C \in Ob(\mathcal{C})$, subject to following relations:

1. $[C] = [C']$ if there exists a w.e. $C \xrightarrow{\sim} C'$.
2. $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

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## Remarks

These relations imply:

1. $[0] = 0$.
2. $[B \coprod C] = [B] + [C]$.
3. $[B \bigcup_A C] = [B] + [C] - [A]$.
4. $[B/A] = [B] - [A]$ since, $B/A = B \bigcup_A 0$.

- From the $S_\bullet$-construction, we can have for following:

$$S_\bullet w\mathcal{C} = \{[n] \mapsto Ob(S_n w\mathcal{C})\} \in \mathbf{sSet}.$$

  So, we can have the loop space of the geometric realization:

$$K(\mathcal{C}) := \Omega|S_\bullet w\mathcal{C}|.$$

- Hence, we have:

$$\pi_i(K(\mathcal{C})) = \pi_i(\Omega|S_\bullet w\mathcal{C}|) \cong \pi_{i+1}(|S_\bullet w\mathcal{C}|) \stackrel{\text{def}}{=} \pi_{i+1}(S_\bullet w\mathcal{C}).$$

# Nerve of a category

Nerve of a small category $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$.

- $N_0(\mathcal{C})$ = 0-cells = $Ob(\mathcal{C})$:

$$\bullet A$$

- $N_1(\mathcal{C})$ = 1-cells = Morphisms of $\mathcal{C}$:

$$A_1 \xrightarrow{f} A_2$$

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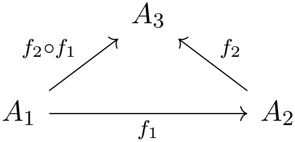
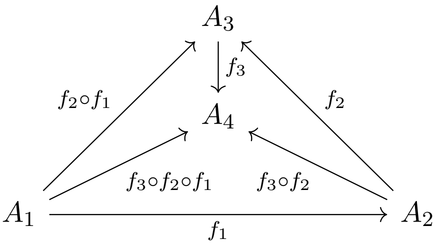
- $N_2(\mathcal{C})$ = 2-cells = A pair of composable morphisms in $\mathcal{C}$:



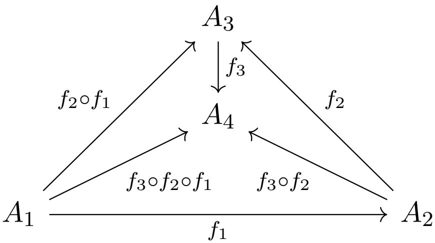

i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$.

- $N_3(\mathcal{C})$ = 3-cells = A triplet of composable morphisms in $\mathcal{C}$:




  i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$.
- and so on.

- $N_3(\mathcal{C})$ = 3-cells = A triplet of composable morphisms in $\mathcal{C}$:




  i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$.
- and so on.
- $d_i : N_k(\mathcal{C}) \to N_{k-1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots \to A_k)$$

$$\downarrow$$

$$(A_1 \to \cdots A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots A_k)$$

- $s_i : N_k(\mathcal{C}) \to N_{k+1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_i \to \cdots \to A_k) \mapsto (A_1 \to \cdots A_i \xrightarrow{\text{id}} A_i \to \cdots A_k).$$

- We define a construction for a Waldhausen category $\mathcal{C}$, denoted by $T_\bullet\mathcal{C}$.
  Where, $T_n\mathcal{C}$ is generated by $N_p(S_q w\mathcal{C}),\ p+q=n+1$.
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- We define a construction for a Waldhausen category $\mathcal{C}$, denoted by $T_\bullet \mathcal{C}$.
  Where, $T_n\mathcal{C}$ is generated by $N_p(S_q w\mathcal{C}),\ p+q=n+1$.
  Here, $w$ stands for considering weak equivalences.
- So, $N_p(S_q w\mathcal{C}) \in \mathbf{s^2Set}$. Up on taking its anti-diagonal (via a w.e. called Artin-Mazur map) becomes a **sSet**.

$$N_pS_qw\mathcal{C} \longmapsto d(N_pS_qw\mathcal{C}) \xmapsto{\text{Artin-Mazur}} T(N_pS_qw\mathcal{C})$$

- Since it is known that $Ob(S_\bullet w\mathcal{C}) \xrightarrow{\sim} d(N_p(S_q w\mathcal{C}))$, the two simplicial sets $Ob(S_\bullet w\mathcal{C})$ and $T_\bullet\mathcal{C}$ are weakly equivalent, so they have same homotopy groups.

# Examples of cells

## Example 11

Given a Waldhausen category $\mathcal{C}$, :
$T_0(\mathcal{C})^a$ consists of:

A

Figure 1: $N_0(S_1 w\mathcal{C})$

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Given a Waldhausen category $\mathcal{C}$, :
$T_0(\mathcal{C})$[a] consists of:

$$A$$

Figure 1: $N_0(S_1 w\mathcal{C})$

Similarly, for the 1-type:
$T_1(\mathcal{C})$ consists of:

$$A_0 \xrightarrow{\sim} A_1$$

Figure 2: $N_1(S_1 w\mathcal{C})$

$$A \rightarrowtail B \twoheadrightarrow C$$

Figure 3: $N_0(S_2 w\mathcal{C})$

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

## Example 12

Again, similarly, for the 2-type:
$T_2(\mathcal{C})$ consists of:

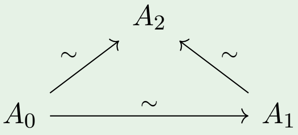


Figure 4: $N_2(S_1 w\mathcal{C})$

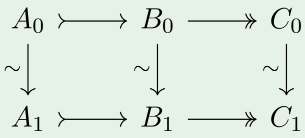


Figure 5: $N_1(S_2 w\mathcal{C})$

## Example 12

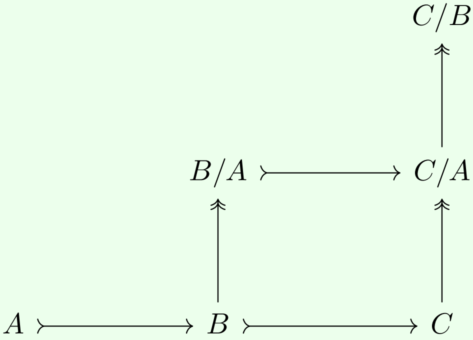


Figure 6: $N_0(S_3 w\mathcal{C})$

# Approximation of a sSet by $n$-types

### Definition 13

$n$-type is the full subcategory of $\text{Top}^*/\cong$ (i.e., pointed topological spaces up to homotopy equivalence) consisting of connected CW-spaces $Y$ with $\pi_i(Y) = 0$ for $i > n$. To appendix

## Fact 14

*For a connected CW-complex $X$, one can construct a sequence of spaces $P_nX$ such that $\pi_i(P_nX) \cong \pi_i(X)$ for $i \leq n$, and $\pi_i(P_nX) = 0$ for $i > n$, and for $i_n : X \to P_nX$, and $j_n : P_nX \to P_{n-1}X$ we have $j_n \circ i_n = i_{n-1}$ for all $n \geq 1$.*

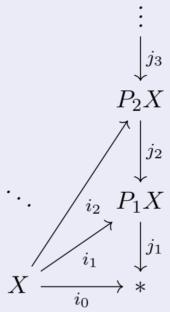


Figure 7: *Postnikov tower*

*This commutative diagram is called a Postnikov tower of $X$, the n-type spaces $P_nX$ are called truncations of $X$.*

# Postnikov tower of a **sSet**

- If $X \in \mathbf{sSet}$, $X$ is fibrant, then $P_n X = Cosk_n(X)$, the tower of Coskeletons via Kan extensions.

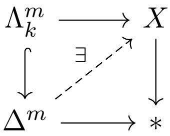


Figure 8: Fibrant object $X$ in **sSet**

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  - The lift exists for each $m, k \in \mathbb{N}$, $k < m$.

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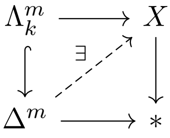


Figure 8: Fibrant object $X$ in **sSet**

  - $\Lambda_k^m$ is a horn.
  - The lift exists for each $m, k \in \mathbb{N}$, $k < m$.
- In general, if $X$ is not fibrant, we can use a fibrant replacement $X \to RX$ where $P_n(X) = Cosk_n(RX)$.
- In general, the $S_\bullet$-construction is not fibrant, so we work with a different (algebraic) model.

# Models for $n$-types: $n = 0, 1$

We want an algebraic model for the types in the Postnikov tower:

$$\begin{array}{ccc} & & T_1\mathcal{C} \\ & \nearrow & \downarrow \\ \mathcal{C} & \longrightarrow & T_0\mathcal{C} \end{array}$$

- $n = 0$: Group, a fundamental group.
- $n = 1$: $D_*^{(1)}(\mathcal{C})$: $D_1^{(1)}(\mathcal{C}) \xrightarrow{\partial} D_0^{(1)}(\mathcal{C})$, a SQuad.

# Models for 1-type

- It is known that a stable quadratic module (SQuad)[1] is 1-type, so we construct a SQuad for a given Waldhausen category $\mathcal{C}$.

[1] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

## Definition 15

A stable quadratic module $C_*$ is a commutative diagram of group homomorphisms To appendix

$$\begin{array}{ccc} C_0^{ab} \otimes C_0^{ab} & & \\ \downarrow w & \searrow^{\text{commutator}} & \\ C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

such that given $c_i, d_i \in C_i, i = 0, 1,$

1. $w(\{\partial(c_1)\} \otimes \{\partial(d_1)\}) = [d_1, c_1] = d_1^{-1} c_1^{-1} d_1 c_1,$
2. $w(\{c_0\} \otimes \{d_0\} + \{d_0\} \otimes \{c_0\}) = 0.$ (The stability condition).

$$C_0 \to C_0^{ab}$$

$$x \mapsto \{x\}$$

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## Remark

The homotopy groups of $C_*$ are:

- $\pi_0(C_*) = \text{Coker}\partial$,
- $\pi_1(C_*) = \text{Ker}\partial$.

# 1-type of a Waldhausen category

$$U : \mathbf{SQuad} \xrightarrow{\text{Forget}} \mathbf{Set} \times \mathbf{Set}$$

$$C_* \mapsto (C_0, C_1).$$

The functor $U$ has a left adjoint $F$, and a SQuad $F(E_0, E_1)$ is called free stable quadratic module on the sets $E_0$ and $E_1$.

**Fact 16**

*Given a Waldhausen category $\mathcal{C}$, we can define a corresponding SQuad $F(T_0(\mathcal{C}), T_1(\mathcal{C}))$[a], where $T_0(\mathcal{C})$, $T_1(\mathcal{C})$ come from example 11, 12.*

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

# Detailed SQuad structure for Fact 16

- The generators for dimension 0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for dimension 1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.

# Detailed SQuad structure for Fact 16

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  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- such that the following relations hold (i.e., we define $\partial, w$):
  - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
  - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - $[0] = 0$.
  - $[A \xrightarrow{id} A] = 0$.
  - $[A \xrightarrow{id} A \twoheadrightarrow 0] = 0, [0 \rightarrowtail A \xrightarrow{id} A] = 0$.
  - For any composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

- For any $A, B \in Ob(\mathcal{C})$, define the $w$ as follows:

$$w([A] \otimes [B]) := \langle [A], [B] \rangle$$

$$=$$

$$-[\ B \overset{i_2}{\rightarrowtail} A \coprod B \overset{p_2}{\twoheadrightarrow} A\ ] + [\ A \overset{i_1}{\rightarrowtail} A \coprod B \overset{p_1}{\twoheadrightarrow} B\ ].$$

Here,

$$A \underset{p_2}{\overset{i_1}{\rightleftarrows}} A \coprod B \underset{i_2}{\overset{p_1}{\rightleftarrows}} B$$

are natural inclusions and projections of a coproduct in $\mathcal{C}$.

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are natural inclusions and projections of a coproduct in $\mathcal{C}$.

- For any commutative diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

we have

$$[A_0 \xrightarrow{\sim} A_1] + [B_0/A_0 \xrightarrow{\sim} B_1/A_1] + \langle [A], -[B_1/A_1] + [B_0/A_0] \rangle$$

$$=$$

$$-[A_1 \rightarrowtail B_1 \twoheadrightarrow B_1/A_1] + [B_0 \xrightarrow{\sim} B_1] + [A_0 \rightarrowtail B_0 \twoheadrightarrow B_0/A_0].$$

- For any commutative diagram consisting of cofiber sequences in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow\!\!\!\!\uparrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow\!\!\!\!\uparrow & & \uparrow\!\!\!\!\uparrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$

we have,

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

$$=$$

$$[\mathrm{A} \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle.$$

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & S/K \\
 & & & & & & & & & & \uparrow \\
 & & & & & & & & K/N & \rightarrowtail & S/N \\
 & & & & & & & & \uparrow & & \uparrow \\
 & & & & & & N/A & \rightarrowtail & K/A & \rightarrowtail & S/A \\
 & & & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & A/H & \rightarrowtail & N/H & \rightarrowtail & K/H & \rightarrowtail & S/H \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & H/T & \rightarrowtail & A/T & \rightarrowtail & N/T & \rightarrowtail & K/T & \rightarrowtail & S/T \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
T & \rightarrowtail & H & \rightarrowtail & A & \rightarrowtail & N & \rightarrowtail & K & \rightarrowtail & S
\end{array}
$$

# References I

[1] Fernando Muro and Andrew Tonks. “The 1-type of a Waldhausen K-theory spectrum”. In: *Advances in Mathematics 216* (2007), pp. 179–183.

[2] W. G. Dwyer and J. Spaliński. “Homotopy theories and model categories”. In: *Handbook of algebraic topology*. North-Holland, Amsterdam, 1995, pp. 73–126. DOI: 10.1016/B978-044481779-2/50003-1. URL: https://doi.org/10.1016/B978-044481779-2/50003-1.

[3] Hans-Joachim Baues. “Combinatorial Homotopy and 4-Dimensional Complexes”. In: *Walter de Gruyter* (1991), pp. 171–177.

# References II


[4] Friedhelm Waldhausen. "Algebraic $K$-theory of spaces". In: *Algebraic and geometric topology (New Brunswick, N.J., 1983).* Vol. 1126. Lecture Notes in Math. Springer, Berlin, 1985, pp. 318–419. DOI: 10.1007/BFb0074449. URL: https://doi.org/10.1007/BFb0074449.

[5] W. G. Dwyer, D. M. Kan, and J. H. Smith. "An obstruction theory for simplicial categories". In: *Nederl. Akad. Wetensch. Indag. Math.* 48.2 (1986), pp. 153–161. ISSN: 0019-3577.

# Coskeletons as a Postnikov decomposition[2]

- Given any $X \in \mathbf{sSet}$, we can have a truncation functor for each $n \in \mathbb{N}$
$$tr_n : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}.$$
- Then by Kan extension we have the following functors:
$$\mathbf{sSet} \underset{\overset{\longleftarrow}{cosk_n}}{\overset{\overset{sk_n}{\longleftarrow}}{\xrightarrow{tr_n}}} \mathbf{sSet}_{\leq n}$$
such that $sk_n \dashv tr_n \dashv cosk_n$.
- Now consider,
$$Sk_n := sk_n \circ tr_n : \mathbf{sSet} \to \mathbf{sSet},$$
$$Cosk_n := cosk_n \circ tr_n : \mathbf{sSet} \to \mathbf{sSet}.$$
Then $Sk_n \dashv Cosk_n$.

[2] W. G. Dwyer, D. M. Kan, and J. H. Smith. "An obstruction theory for simplicial categories". In: *Nederl. Akad. Wetensch. Indag. Math.* 48.2 (1986), pp. 153–161. ISSN: 0019-3577.

- They also satisfy the following properties:
  - $(Cosk_n X)_k \cong sSet(\Delta^k, Cosk_n X) \cong sSet(Sk_n \Delta^k, X)$.
  - If $k \leq n$: $Sk_n \Delta^k = \Delta^k$, $(Cosk_n X)_k = X_k$.
  - If $k = n+1$:
    $(Cosk_n X)_{n+1} \cong sSet(Sk_n \Delta^{n+1}, X) \cong sSet(\partial \Delta^{n+1}, X) = 0$.
- $Cosk_n$ is a right adjoint, so it preserves fibrant object. So, when $X$ is fibrant, then so is $Cosk_n X$ and its homotopy groups are trivial in dimension $\geq n$.
- Hence, the sequence:
  $\mathrm{X} = \lim\limits_{\leftarrow}\big(\cdots \to Cosk_{n+1}(X) \to Cosk_n(X) \to Cosk_{n-1}(X) \to \cdots \to *\big)$
  is up to homotopy, a Postnikov decomposition of $X$.

# Serre cofibrations

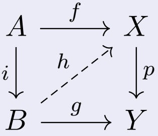
- In the category of topological spaces, a map $f : X \to Y$ is called a Serre fibration, if for each CW-complex $A$, the map $f$ has the RLP w.r.t. the inclusion $A \times \{0\} \to A \times [0,1]$:

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

- A map $f$ is called a Serre cofibration if it has the LLP w.r.t. acyclic fibrations.

## Definition 17

A map $i : A \to B$ is said to have the left lifting property (LLP)[a] with respect to another map $p : X \to Y$ and $p$ is said to have the right lifting property (RLP) with respect to $i$ if a lift $h : B \to X$ exists for any of the commutative diagram of the following form: Back to main




[a] W. G. Dwyer and J. Spaliński. "Homotopy theories and model categories". In: *Handbook of algebraic topology*. North-Holland, Amsterdam, 1995, pp. 73–126. DOI: 10.1016/B978-044481779-2/50003-1. URL: https://doi.org/10.1016/B978-044481779-2/50003-1.

## Fact 18

*The fibrations (in the sense of Model category) are the maps that have the RLP with respect to acyclic cofibrations (i.e., cofibrations that are also w.e.).*

Definition 19

An object $A$ is called fibrant if $A \to 0$ is a fibration.

- Consider

$$U\colon \mathbf{SQuad} \xrightarrow{\text{Forget}} \mathbf{Set} \times \mathbf{Set}$$

$$C_* \mapsto (C_0, C_1).$$

The functor $U$ has a left adjoint $F$, and a SQuad $F(E_0, E_1)$ is called free stable quadratic module[1] on the sets $E_0$ and $E_1$.

- Given a set $E$,
  - denote the free generated with basis $E$ by $\langle E \rangle$,
  - free abelian group with basis $E$ by $\langle E \rangle^{ab}$,
  - free group of nilpotency class 2 with basis $E$ by $\langle E \rangle^{nil}$ (i.e., the quotient of $\langle E \rangle$ by triple commutators),
- Given an abelian group $A$,
  - denote the quotient of $A \otimes A$ by $a \otimes b + b \otimes a, a, b \in A$ by $\hat{\otimes}^2 A$.

- Given a pair of sets $E_0$ and $E_1$,
  - write $E_0 \cup \partial E_1$ for the set whose elements are the symbols $e_0$ and $\partial e_1$ for each $e_0 \in E_0, e_1 \in E_1$.

Then we can define the free SQuad by considering:

  - $F(E_0, E_1)_0 = \langle E_0 \cup \partial E_1 \rangle^{nil}$,
  - $F(E_0, E_1)_1 = \hat{\otimes}^2 \langle E \rangle^{ab} \times \langle E_0 \times E_1 \rangle^{ab} \times \langle E_1 \rangle^{nil}$.

# Simplicial Set

A simplicial set $X \in \mathbf{sSet}$ is

- for each $n \in \mathbb{N}$ a set $X_n \in \mathbf{Set}$ (the set of $n$-simplices),
- for each injective map $\partial_i : [n1]ß[n]$ of totally ordered sets ($[n] \colon = (0 < 1 < \cdots < n)$,
- a function $d_i : X_n \to X_{n1}$ (the $i^{\text{th}}$ face map on $n$-simplices) ($n > 0$ and $0in$),
- for each surjective map $\sigma_i : [n+1] \to [n]$ of totally ordered sets,
- a function $s_i : X_n \to X_{n+1}$ (the $i^{\text{th}}$ degeneracy map on $n$-simplices) ($n \geq 0$ and $0 \leq i \leq n$),
- such that these functions satisfy the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{when } i < j, \\ 1, & \text{when } i = j, j+1, \\ s_j d_{i-1}, & \text{when } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ when } i \leq j$$

# Definition of a Quad[3]

**Definition 20**

A pre-crossed module $G_*$ is a equivariant $G_0$-group homomorphism $\partial : G_1 \to G_0$, where $G_0$ acts on itself by conjugation.

[3] Hans-Joachim Baues. "Combinatorial Homotopy and 4-Dimensional Complexes". In: *Walter de Gruyter* (1991), pp. 171–177.

## Definition 21

A quadratic module $(w, \delta, \partial)$ is a complex of $G_0$-groups

$$\begin{array}{ccccc} & & (G_1^{\mathrm{cr}})^{\mathrm{ab}} \times (G_1^{\mathrm{cr}})^{\mathrm{ab}} & & \\ & \swarrow^{w} & \downarrow \mathrm{w} & & \\ G_2 & \xrightarrow{\delta} & G_1 & \xrightarrow{\partial} & G_0 \end{array}$$

where, $G_1^{\mathrm{cr}}$ is a group such that the pre-cross module $\partial : G_1 \to G_0$ becomes a crossed module $\partial : G_1^{\mathrm{cr}} \to G_0$.
such that

- $\partial : G_1 \to G_0$ is a $nil(2)$-module.
- $\partial\delta = 0$, $\delta w = \mathrm{w} =$ Peiffer commutator map:
  $$\mathrm{w}(x \otimes y) = -x - y + x + y^{\partial x}$$
- All homomorphisms are equivariant with respect to the action of $G_0$
- $f^{\partial x} = f + w(\{\partial f\} \otimes \{x\} + \{x\} + \{\partial f\})$ for all $f \in G_2, x \in G_1$.
- $w(\{\partial a\} \otimes \{\partial b\}) = [a, b] = -a - b + a + b$.

## Remark

- Putting $G_0 = 0$ in the definition above gives us the Definition 15.
- Homotopy groups of the quadratic module $\sigma = (w, \delta, \partial)$ can be defined as:
  - $\pi_1(\sigma) = \mathrm{Coker}(\partial)$,
  - $\pi_2(\sigma) = \mathrm{Ker}(\partial)/\mathrm{Im}(\delta)$,
  - $\pi_3(\sigma) = \mathrm{Ker}(\delta)$.
- From Definition 15, we can conclude that $C_0$ and $C_1$ are groups of nilpotency class 2.
  - Given $x, y, z \in C_0$, we have:

    $$[x, [y, z]] = \partial w(\{[y, z]\} \otimes \{x\}) = \partial w(0 \otimes \{x\}) = 0.$$

  - Similarly, given $f, g, h \in C_1$ we have:
    $[f, [g, h]] = w(\{\partial([g, h])\} \otimes \{\partial(f)\}) = w(\{[\partial(g), \partial(h)]\} \otimes \{\partial(f)\}) = w(0 \otimes \{\partial(f)\}) = 0.$