Monoidal 2-Categories and K-theory of Waldhausen Categories

Milind Gunjal

Department of Mathematics
Florida State University

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This thesis consists of two parts:

1. Connected 3-type of K-theory of Waldhausen Categories.
2. Symmetric Monoidal Bicategories and Biextensions.[1]

[1] Ettore Aldrovandi and Milind Gunjal. "Symmetric Monoidal Bicategories and Biextensions". In: *https://arxiv.org/abs/2411.10530* (2024).

Waldhausen Category

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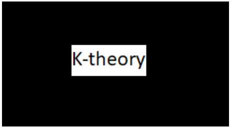

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Spectrum $\{X_i\}_{i\geq 0}$ with interesting homotopy groups

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n-types (in terms of Algebraic Models)

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Stabilize the Algebraic Models

Definition of Waldhausen Categories

A Waldhausen category[a] $\mathcal{C}$ is a category with a zero object, 0 equipped with two classes of morphisms: weak equivalences (WE) and cofibrations (CO) such that

- $\mathrm{Iso}(\mathcal{C}) \subseteq \mathrm{WE}(\mathcal{C}) \cap \mathrm{CO}(\mathcal{C})$.
- $0 \to X \in \mathrm{CO}(\mathcal{C})$ for all $X \in \mathrm{Ob}(\mathcal{C})$.
- If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \sqcup_A C$ of these two maps exists in $\mathcal{C}$ and $C \rightarrowtail B \sqcup_A C$ is a cofibration.

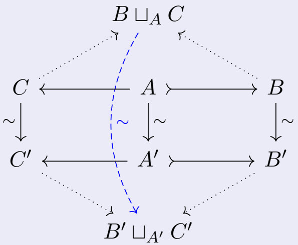
$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \sqcup_A C \end{array}$$

  - In particular, coproduct of two objects $B \sqcup C = B \sqcup_0 C$ exists.

[a] Friedhelm Waldhausen. "Algebraic K-theory of spaces". In: vol. 1126. Lecture Notes in Math. Springer, Berlin, 1985, pp. 318–419.

Definition of Waldhausen Categories

- Gluing axiom:



- Extension axiom:

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

Examples of Waldhausen Categories

1. The category of finitely generated projective R-Mod, for any commutative ring R.
   - Injective maps (CO).
   - Isomorphisms (WE).
2. An exact category.
   - Monomorphisms (CO).
   - Isomorphisms (WE).
3. Category $\mathcal{R}(X)$ of spaces that retract to X.
   - Serre cofibrations (CO). To apdx
   - Maps that induce isomorphisms for chosen homology theory (WE).
4. The category of finite sets.
   - Inclusions (CO).
   - Isomorphisms (WE).

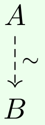
- A spectrum is a sequence of pointed spaces $\{X_i\}_{i\geq 0}$ with the structure maps $\Sigma X_i \to X_{i+1}$.
- The K-theory spectrum is an Ω-spectrum, i.e., $X_n \simeq \Omega X_{n+1}$. So, studying the base space is enough.
- The K-theory space of a Waldhausen category $\mathcal{C}$

$$K(\mathcal{C}) = \Omega \left| NwS_0\mathcal{C} \underset{d_i}{\overset{s_0}{\leftleftarrows}} NwS_1\mathcal{C} \underset{d_i}{\overset{s_i}{\Lleftarrow}} NwS_2\mathcal{C} \underset{d_i}{\overset{s_i}{\Lleftarrow}} NwS_3\mathcal{C} \underset{d_i}{\overset{s_i}{\Lleftarrow}} \cdots \right|$$

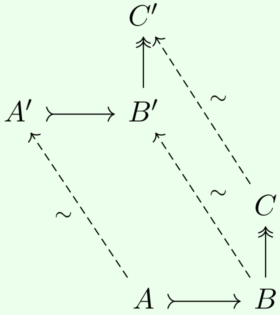
  - N is the Nerve.
  - Ω is the loop space.

$S_\bullet$-Construction

- $wS_0\mathcal{C}$: Trivial category (One object, its identity morphism)
- $wS_1\mathcal{C}$: $w\mathcal{C}$



- $wS_2\mathcal{C}$:



$S_\bullet$-Construction

- $wS_3\mathcal{C}$:

$$
\begin{array}{ccccc}
 & & & & F \\
 & & & & \uparrow\!\!\uparrow \\
 & & D & \rightarrowtail & E \\
 & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
A & \rightarrowtail & B & \rightarrowtail & C
\end{array}
$$

- The K groups of a Waldhausen category $\mathcal{C}$ are

$$K_n(\mathcal{C}) := \pi_n(K(\mathcal{C})).$$

Homotopy groups of a Spectrum

For a spectrum $\mathbb{X} = \{X_n\}_{n\geq 0}$, the following sequence exists due to structure maps $\Sigma X_n \to X_{n+1}$.

$$\pi_i(X_n) \longrightarrow \pi_{i+1}(X_{n+1}) \longrightarrow \pi_{i+2}(X_{n+2}) \longrightarrow \cdots$$

The i^{th} homotopy group of $\mathbb{X}$ is:

$$\pi_i^s(\mathbb{X}) := \varinjlim_{k} \pi_{i+k}(X_k).$$

- Since $K(\mathcal{C})$ is an Ω-spectrum

$$K_n(\mathcal{C}) = \pi_n^s(K(\mathcal{C})).$$

Waldhausen Category

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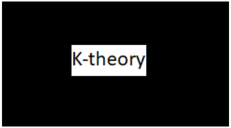

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Spectrum $\{X_i\}$ with interesting homotopy groups

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n-types (in terms of Algebraic Models)

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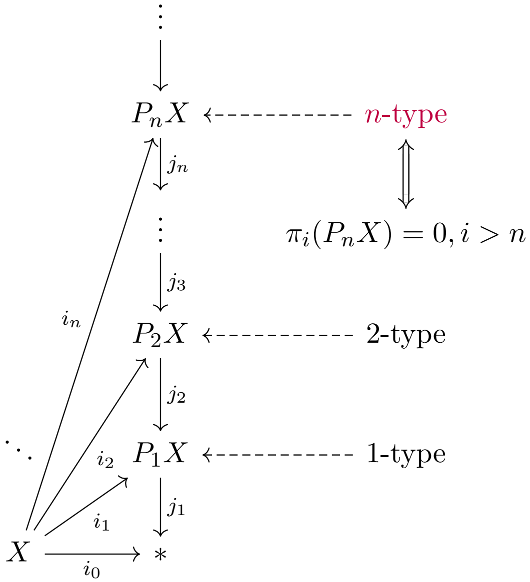
Stabilize the Algebraic Models



Postnikov Tower for a connected pointed space X.

Theorem (Homotopy Hypothesis (Grothendieck))

By taking classifying spaces and fundamental n-groupoids, there is an equivalence between the theory of weak n-goupoids and that of homotopy n-types.

Connected types	Algebraic model		Categorical model
1-type	Group	G	$G \rightrightarrows *$
2-type	Crossed Module	$G_1 \xrightarrow{\partial} G_0$	$\Gamma(G_*) \rightrightarrows *$
3-type	Quadratic Module	$H_2 \xrightarrow{\delta} H_1 \xrightarrow{\partial} H_0$	$\Gamma(H_*) \rightrightarrows *$

- Muro and Tonks construct a stable crossed module

$$G_1 \xrightarrow{\partial} G_0$$

explicitly using generators and relations.[2]
  - The generators for G_0 are:
    - $[A]$ for any $A \in \mathrm{Ob}(\mathcal{C})$.
  - The generators for G_1 are:
    - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
    - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
  - Boundary maps are:
    - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
    - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - Satisfying certain relations.
- $\pi_1(G_*) = \mathrm{coker}\ \partial = K_0(\mathcal{C})$.
- $\pi_2(G_*) = \ker \partial = K_1(\mathcal{C})$.

[2] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Definition of Quadratic Modules

A quadratic module is a diagram

$$\begin{array}{ccccc} & & (M^{\mathrm{cr}})^{\mathrm{ab}} \otimes (M^{\mathrm{cr}})^{\mathrm{ab}} & & \\ & {}^{\omega}\swarrow & \downarrow w & & \\ L & \xrightarrow{\quad\delta\quad} & M & \xrightarrow{\quad\partial\quad} & N \end{array}$$

of homomorphisms between groups such that

1. $\partial \colon M \longrightarrow N$ is a nil(2)-module with Peiffer commutator map w.

$$w(x \otimes y) = -x - y + x + y^{\partial x}$$

2. $\partial \circ \delta = 0$.
3. L is an N-group and δ and ∂ are N-equivariant.
4. The action of N on L satisfies the following:

$$l^{\partial m} = l + \omega(\{\delta l\} \otimes \{m\} + \{m\} \otimes \{\delta l\}) \text{ for all } l \in L, m \in M.$$

- A free quadratic module is a quadratic module that satisfies a certain universal property.
- Baues[3] defines a functor

$$Q\colon \mathbf{sSet_0} \longrightarrow \mathbf{FreeQuad}$$

  $Q(X_\bullet)$ is defined using bases
  - $d_2\colon \langle X_2\rangle \longrightarrow N$, where $N = \langle X_1\rangle$.
  - $d_3\colon \langle X_3\rangle \longrightarrow M$.

[3] Hans-Joachim Baues. *Combinatorial Homotopy and 4-Dimensional Complexes*. Walter de Gruyter, 1991, pp. 177–187.

- Since the K-theory is a spectrum, the types must be stable, i.e., their homotopy groups simply shift after taking suspension.
- The 2-type is stable due to its construction.
- For the 3-type, we pull the stable structure back along the 2-category that the quadratic module creates.

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Theorem (Stable Homotopy Hypothesis)

Symmetric monoidal structure corresponds to topological stability.[a]

Stable 2-types $\longleftrightarrow$ Symmetric monoidal category $\longleftrightarrow$ Stable Quadratic Module

$\downarrow$ $\quad$ $\downarrow$ $\quad$ $\downarrow$

Stable 3-types $\longleftrightarrow$ Symmetric monoidal 2-category $\longleftrightarrow$ Stable Quadratic 2-Modules

$\downarrow$ $\quad$ $\downarrow$ $\quad$ $\downarrow$

$\vdots$ $\quad$ $\vdots$ $\quad$ $\vdots$

[a] Nick Gurski, Niles Johnson, and Angélica M. Osorno. "The 2-dimensional stable homotopy hypothesis". In: *Journal of Pure and Applied Algebra, Volume 223, Issue 10, 2019* (2019), pp. 4348–4383.

Waldhausen Category

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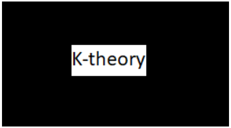

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Stabilize the Algebraic Models

- Recall: $K(\mathcal{C}) = \Omega|NwS_\bullet\mathcal{C}|$.
- We choose codiagonalization functor for geometric realization.

$$T\colon \mathbf{s^2Set_0} \longrightarrow \mathbf{sSet_0}$$

Codiagonalization

Let X be a bisimplicial set.

$$(TX)_n = \{(x_{0,n}, x_{1,n-1}, \cdots, x_{n,0}) \mid d_0^v x_{p,n-p} = d_{p+1}^h x_{p+1} \forall p\}.$$

Here, d_0^v and d_{p+1}^h are vertical and horizontal face maps, respectively.

  - T gives simpler cells compared to other possible functors.
  - T preserves products.

Cells of $T(K(\mathcal{C}))$

For $A, X, Y, U, V, W \in \text{Ob}(\mathcal{C})$

- X_1: A
- X_2: $\begin{array}{ccc} X & \rightarrowtail & Y \\ {\scriptstyle\sim}\uparrow & & \\ A & & \end{array}$
- X_3: $\begin{array}{ccccc} U & \rightarrowtail & V & \rightarrowtail & W \\ {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & \\ X & \rightarrowtail & Y & & \\ {\scriptstyle\sim}\uparrow & & & & \\ A & & & & \end{array}$

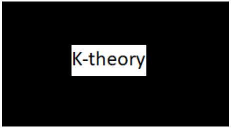
- Construct a free quadratic module from $T(K(\mathcal{C}))$.
- Bases for $(Q(T(K(\mathcal{C}))))_n$ are the non-degenerate elements of $(T(K(\mathcal{C})))_n$.
- For $\zeta \in (T(K(\mathcal{C})))_n$, define

$$|\zeta| = \begin{cases} \zeta & \text{if } \zeta \text{ is non-degenerate,} \\ 0 & \text{otherwise} \end{cases}$$

- $$d_2\left(\left|\begin{array}{ccc} X & \rightarrowtail & Y \\ \sim\uparrow & & \\ A & & \end{array}\right|\right) = -|Y| + |Y/X| + |A|.$$
- $$d_3\left(\left|\begin{array}{ccccc} U & \rightarrowtail & V & \rightarrowtail & W \\ \sim\uparrow & & \sim\uparrow & & \\ X & \rightarrowtail & Y & & \\ \sim\uparrow & & & & \\ A & & & & \end{array}\right|\right) = \left|\begin{array}{ccc} U & \rightarrowtail & W \\ \sim\uparrow & & \\ A & & \end{array}\right| + \left|\begin{array}{ccc} V/U & \rightarrowtail & W/U \\ \sim\uparrow & & \\ Y/X & & \end{array}\right|^{|A|} - \left|\begin{array}{ccc} X & \rightarrowtail & Y \\ \sim\uparrow & & \\ A & & \end{array}\right| - \left|\begin{array}{ccc} V & \rightarrowtail & W \\ \sim\uparrow & & \\ Y & & \end{array}\right|.$$

Waldhausen Category

$\downarrow$



$\downarrow$

Spectrum $\{X_i\}$ with interesting homotopy groups

$\downarrow$

n-types (in terms of Algebraic Models)

$\downarrow$

Stabilize the Algebraic Models

We now construct Γ: **Quad** $\longrightarrow$ **SM 2-Cat**

- Given a Quad σ:

$$L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

- $\mathrm{Ob}(\Gamma(\sigma)) = N$.

$$x_0 \in N.$$

- $1\text{-}\mathrm{Mor}(\Gamma(\sigma)) = M \ltimes N$.

$$x_0 \xrightarrow{f} x_1 \quad \text{such that } x_1 = x_0 \cdot \partial(f).$$

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- Given a Quad σ:

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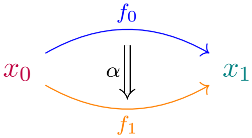
- 1-Mor$(\Gamma(\sigma)) = M \ltimes N$.

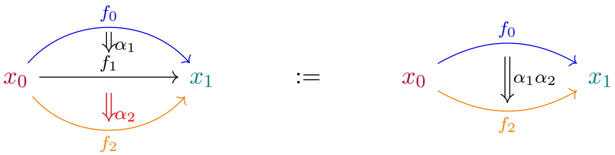
$$x_0 \xrightarrow{f} x_1 \text{ such that } x_1 = x_0 \cdot \partial(f).$$

- 2-Mor$(\Gamma(\sigma)) = L \ltimes M \ltimes N$.

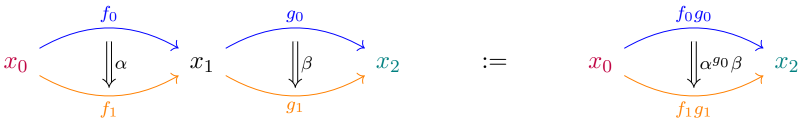


Such that $f_1 = f_0 \cdot \delta(\alpha)$.

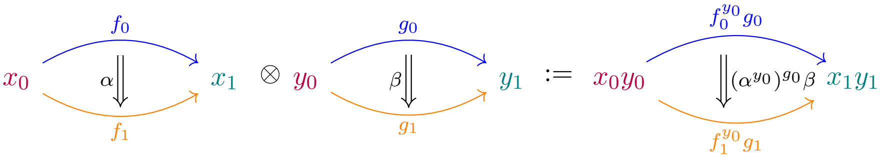
- Vertical composition:



- Horizontal composition:



- Monoidal structure:



Components of a Symmetric Monoidal 2-Category are:

- A 2-Category.
- Monoidal structure ($\otimes$) on the 2-Category.

Components of a Symmetric Monoidal 2-Category are:

- A 2-Category.
- Monoidal structure ($\otimes$) on the 2-Category.
  - Braiding (β) on the monoidal structure.
  - Left ($\eta_{-|--}$) and right ($\eta_{--|-}$) hexagonators.
  - Syllepsis (γ).
    - Symmetry axiom.

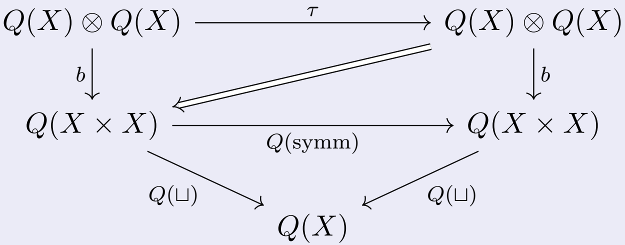
★ Pull back the symmetric structure to get a stabilized Quad.

★ This gives us an algebraic model for a connected 3-type.

Main Theorem [Aldrovandi, G.]

Let $\mathcal{C}$ be a Waldhausen category.

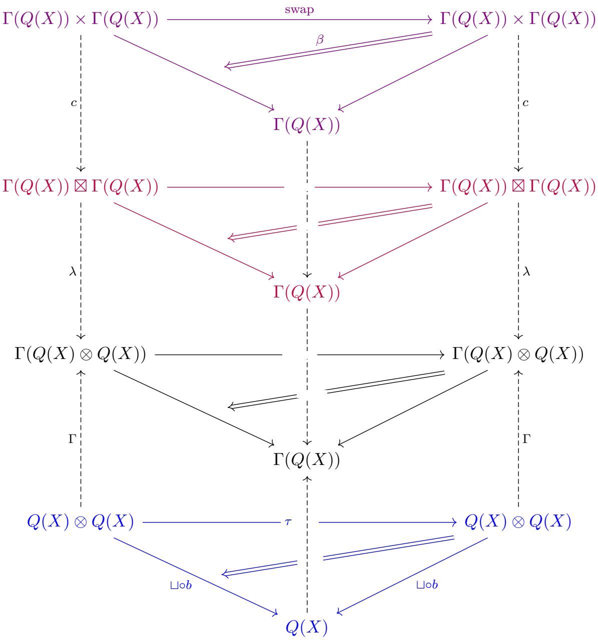
1. $Q(T(K(\mathcal{C})))$ is a connected 3-type of $\mathcal{C}$.
2. The braiding of the SM 2-Category is given by the following diagram in **FreeQuad**.



- $\tau(e \otimes f) = (-1)^{|e||f|} f \otimes e$.
- b is the shuffle map. (See [Ton03].)
- $X = T(N(wS_\bullet \mathcal{C})) \in \mathbf{sSet_0}$.

- We need to check that the following diagram commutes.

$$
\begin{array}{ccc}
(Q(X) \otimes Q(X))_1 & \xrightarrow{\sqcup \circ b} & (Q(X))_1 \\
d_2 \uparrow & & \uparrow d_2 \\
(Q(X) \otimes Q(X))_2 & \xrightarrow{\sqcup \circ b} & (Q(X))_2 \\
d_3 \uparrow & & \uparrow d_3 \\
(Q(X) \otimes Q(X))_3 & \xrightarrow{\sqcup \circ b} & (Q(X))_3
\end{array}
$$


$\Gamma(Q(X)) \times \Gamma(Q(X))$
swap
$\Gamma(Q(X)) \times \Gamma(Q(X))$
β
c
c
$\Gamma(Q(X))$
$\Gamma(Q(X)) \boxtimes \Gamma(Q(X))$
$\Gamma(Q(X)) \boxtimes \Gamma(Q(X))$
λ
λ
$\Gamma(Q(X))$
$\Gamma(Q(X) \otimes Q(X))$
$\Gamma(Q(X) \otimes Q(X))$
Γ
Γ
$\Gamma(Q(X))$
$Q(X) \otimes Q(X)$
τ
$Q(X) \otimes Q(X)$
$\sqcup \circ b$
$\sqcup \circ b$
$Q(X)$

- Let $Q, Q' \in$ **FreeQuad**; then there exists a 2-functor

$$\lambda_{Q,Q'} \colon \Gamma(Q) \boxtimes \Gamma(Q') \longrightarrow \Gamma(Q \otimes Q').$$

  - $\lambda^0_{Q,Q'}(q, q') := q \otimes * + * \otimes q'$.
  - $\lambda^1_{Q,Q'}(f \boxtimes 1_b) := (f \otimes *)^{(*\otimes b)}$.
  - $\lambda^1_{Q,Q'}(1_a \boxtimes g) := * \otimes g$.

$$\begin{array}{ccc} (q_0, q_0') & \xrightarrow{f \boxtimes 1} & (q_1, q_0') \\ {\scriptstyle 1\boxtimes g}\downarrow & \swarrow\!\!\!\!\swarrow \; {\scriptstyle \Sigma_{f,g}} & \downarrow{\scriptstyle 1\boxtimes g} \\ (q_0, q_1') & \xrightarrow[f \boxtimes 1]{} & (q_1, q_1') \end{array}$$

  - $\lambda^2_{Q,Q'}\left(\Sigma_{f,g}\right) := \omega\left(\{* \otimes g\} \otimes \left\{(f \otimes *)^{(*\otimes b)}\right\}\right)$

Symmetric Structure of the SM 2-Cat $\Gamma(Q(T(K(\mathcal{C}))))$

For $Y, Z \in \mathrm{Ob}(\mathcal{C})$, we get the braiding

$$\beta_{|Y|,|Z|} \colon |Z| + |Y| \longrightarrow |Y| + |Z|$$

$$\beta_{|Y|,|Z|} = -\left| \begin{array}{ccc} Z & \rightarrowtail & Y \sqcup Z \\ \sim\uparrow & & \\ Z & & \end{array} \right| + \left| \begin{array}{ccc} Y & \rightarrowtail & Y \sqcup Z \\ \sim\uparrow & & \\ Y & & \end{array} \right|$$

We get this from the symmetry in Waldhausen categories and the shuffle map

$$b \colon Q(U) \otimes Q(V) \to Q(U \times V) \text{ here } U, V \in \mathbf{sSet_0}$$

$$b(x \otimes y) = -(s_0 x, s_1 y) + (s_1 x, s_0 y),\ x \in (Q(U))_1, y \in (Q(V))_1$$

The other project

- We analyze (symmetric) monoidal bi-categories using categorical biextensions
  - SM Bicategories are used to study higher algebraic structures like K-theory, operads, etc.
- Let $B \in \mathbf{Grp}$, and $A \in \mathbf{Ab}$, then a group E is an extension of B by A if

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0 \text{ is a short exact sequence}$$

- If $A \to E$ factors through the center of E, it is called a central extension.
- $H^2(B, A) \cong CentrExt(B, A)$.

- Let G be a group, $\mathcal{A}$ be a Picard groupoid. A monoidal category $\mathcal{E}$ is called a categorical extension of G by $\mathcal{A}$ if:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{E} \longrightarrow G \longrightarrow 0$$

- Let $\mathcal{E}$ be a monoidal category, then consider $\mathcal{A} = (\pi_1(\mathcal{E}) \rightrightarrows *)$ and $G = \pi_0(\mathcal{E})$.
  - $\pi_0(\mathcal{E}) = \mathrm{Ob}(\mathcal{E})/\cong$
  - $\pi_1(\mathcal{E}) = \mathrm{Aut}_{\mathcal{E}}(I)$

Theorem

With $\mathcal{E}, G, A$ as above, the monoidal category $\mathcal{E}$ can be classified by $H^2(G, (A \rightrightarrows *)) = H^3(G, A)$.

Theorem

A monoidal bicategory can be classified by $H^3(G, \mathcal{A})$.

- To analyze symmetry, we need extra structure.

- Let $A \in \mathbf{Ab}$, $G, H \in \mathbf{Grp}$. A biextension E, of $G \times H$ by A is an A-torsor[4] over $G \times H$,

$$A \xrightarrow{i} E \xrightarrow{p} G \times H$$

that is a central extension in each variable and has the following maps.

  - Partial composition laws of A-torsors.

$$+_1 : E_{x,y} \wedge^A E_{x',y} \longrightarrow E_{xx',y} \;, x, x' \in G; y \in H$$

$$+_2 : E_{x,y} \wedge^A E_{x,y'} \longrightarrow E_{x,yy'} \;, x \in G; y, y' \in H$$

    They are required to be

    1. associative
    2. compatible with each other, i.e.,

$$(f_{x,y} +_1 f_{x',y}) +_2 (f_{x,y'} +_1 f_{x',y'}) = (f_{x,y} +_2 f_{x,y'}) +_1 (f_{x',y} +_2 f_{x',y'}).$$

[4] A left A-torsor is a non-empty set S equipped with a A-action, $\rho : A \times S \to S$ such that (ρ, π_2): $A \times S \to S \times S$ is an isomorphism.

- For a (symmetric) monoidal category $\mathcal{E}$, Breen considers biextensions of $\pi_0(\mathcal{E}) \times \pi_0(\mathcal{E})$ by $\pi_1(\mathcal{E})$

$$A \xrightarrow{i} E \xrightarrow{p} G \times G$$

where $E_{x,y} = \mathrm{Hom}_{\mathcal{E}}(YX, XY)$ are the A-torsors.

[5] Ettore Aldrovandi and Milind Gunjal. "Symmetric Monoidal Bicategories and Biextensions". In: *https://arxiv.org/abs/2411.10530* (2024).

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$$A \xrightarrow{i} E \xrightarrow{p} G \times G$$

where $E_{x,y} = \mathrm{Hom}_{\mathcal{E}}(YX, XY)$ are the A-torsors.

- We consider categorical biextensions related to a monoidal bicategory $\mathbb{E}$:

$$\mathcal{A} \xrightarrow{i} \mathcal{E} \xrightarrow{p} G \times G$$

  - Partial composition laws of $\mathcal{A}$-torsors $\mathcal{E}_{x,y} = \mathcal{H}om_{\mathbb{E}}(YX, XY)$.

$$+_1 : \mathcal{E}_{x,y} \wedge^{\mathcal{A}} \mathcal{E}_{x',y} \longrightarrow \mathcal{E}_{xx',y} \ , x, x' \in G; y \in H$$

$$+_2 : \mathcal{E}_{x,y} \wedge^{\mathcal{A}} \mathcal{E}_{x,y'} \longrightarrow \mathcal{E}_{x,yy'} \ , x \in G; y, y' \in H$$

- We calculate cocycles explicitly using coherence conditions and classify symmetric monoidal bicategories using the cocycles.[5]

[5] Ettore Aldrovandi and Milind Gunjal. "Symmetric Monoidal Bicategories and Biextensions". In: *https://arxiv.org/abs/2411.10530* (2024).

$$
\begin{array}{ccccccccccc}
T & \rightarrowtail & H & \rightarrowtail & A & \rightarrowtail & N & \rightarrowtail & K & \rightarrowtail & S \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & \\
T & \rightarrowtail & H & \rightarrowtail & A & \rightarrowtail & N & \rightarrowtail & K & & \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & & & \\
T & \rightarrowtail & H & \rightarrowtail & A & \rightarrowtail & N & & & & \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & & & & & \\
T & \rightarrowtail & H & \rightarrowtail & A & & & & & & \\
{\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & & & & & & & \\
T & \rightarrowtail & H & & & & & & & & \\
{\scriptstyle\sim}\uparrow & & & & & & & & & & \\
T & & & & & & & & & &
\end{array}
$$

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- In the category of topological spaces, a map $f : X \to Y$ is called a Serre fibration, if for each CW-complex A, the map f has the RLP w.r.t. the inclusion $A \times \{0\} \to A \times [0,1]$:

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

- A map f is called a Serre cofibration if it has the LLP w.r.t. acyclic fibrations.

Some facts

- Examples of a model category which is not a Waldhausen category: **sSet**
  - It is a model category with monomorphisms as cofibrations, w.e. as the maps that preserve homotopy groups after taking geometric realization
  - But not a Waldhausen category as it does not have the zero object (initial object is the empty set and final object is Δ^0(the one-point sSet)).

Simplicial Set

A simplicial set $X \in \mathbf{sSet}$ is

- for each $n \in \mathbb{N}$ a set $X_n \in \mathbf{Set}$ (the set of n-simplices),
- for each injective map $\partial_i \colon [n-1] \to [n]$ of totally ordered sets ($[n] := (0 < 1 < \cdots < n)$),
- a function $d_i : X_n \to X_{n-1}$ (the i^{th} face map on n-simplices) ($n > 0$ and $0 \leq i \leq n$),
- for each surjective map $\sigma_i : [n+1] \to [n]$ of totally ordered sets,
- a function $s_i : X_n \to X_{n+1}$ (the i^{th} degeneracy map on n-simplices) ($n \geq 0$ and $0 \leq i \leq n$),
- such that these functions satisfy the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{when } i < j, \\ 1, & \text{when } i = j, j+1, \\ s_j d_{i-1}, & \text{when } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ when } i \leq j$$

Nerve of a category

Nerve of a small category $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$.

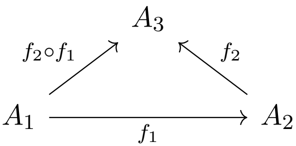
- $N_0(\mathcal{C}) = $ 0-cells $= Ob(\mathcal{C})$:

$$\bullet A$$

- $N_1(\mathcal{C}) = $ 1-cells $=$ Morphisms of $\mathcal{C}$:

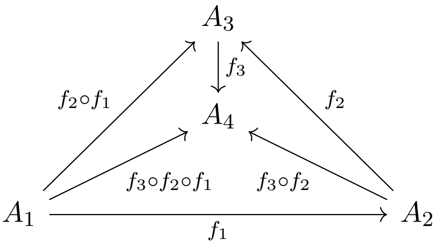
$$A_1 \xrightarrow{f} A_2$$

- $N_2(\mathcal{C}) = $ 2-cells $=$ A pair of composable morphisms in $\mathcal{C}$:



i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$.

- $N_3(\mathcal{C})$ = 3-cells = A triplet of composable morphisms in $\mathcal{C}$:



  i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$.
- and so on.
- $d_i : N_k(\mathcal{C}) \to N_{k-1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots \to A_k)$$
$$\downarrow$$
$$(A_1 \to \cdots A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots A_k)$$

- $s_i : N_k(\mathcal{C}) \to N_{k+1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_i \to \cdots \to A_k) \mapsto (A_1 \to \cdots A_i \xrightarrow{\text{id}} A_i \to \cdots A_k).$$

Definition 1

A Picard groupoid $\mathcal{P}$ is a symmetric monoidal category such that all morphisms are invertible and tensoring with any object X in $\mathcal{P}$ yields an equivalence of categories

$$X \otimes \_ \colon \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Definition 1

A Picard groupoid $\mathcal{P}$ is a symmetric monoidal category such that all morphisms are invertible and tensoring with any object X in $\mathcal{P}$ yields an equivalence of categories

$$X \otimes \_ : \mathcal{P} \xrightarrow{\sim} \mathcal{P}.$$

Definition 2

A Picard 2-groupoid $\mathbb{P}$ is a symmetric monoidal 2-category such that all morphisms are invertible up to 2-morphisms, all 2-morphisms are invertible, and tensoring with any object X in $\mathbb{P}$ yields an equivalence of 2-categories

$$X \otimes \_ : \mathbb{P} \xrightarrow{\sim} \mathbb{P}.$$

Example 3

The prototypical example of a determinant functor on an exact category is the following. Suppose X is a scheme or manifold. Then the rank of a vector bundle E over X is a locally constant function $\mathrm{rk}\ E : X \to \mathbb{Z}$, and we can define a determinant functor $\det : \mathbf{vect}(X) \to \mathbf{Pic}^{\mathbb{Z}}(X)$ as follows:

$$\det(E) = \left(\mathrm{rk}\ E, \wedge_{\mathcal{O}_X}^{\mathrm{rk}\ E} E\right)$$

- Associativity: Given a staircase commutative diagram

$$\Theta : \qquad \begin{array}{ccccc} & & & & C^g \\ & & & & \uparrow \\ & & C^f & \rightarrowtail & C^{gf} \\ & & \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

the following diagram in $\mathcal{P}$ commutes:

$$\begin{array}{ccc} & \det(Z) & \\ \det\left(C^{gf}\right) \otimes \det(X) \xrightarrow{\det(\Delta_g)} \det(Z) & & \det(Z) \xleftarrow{\det\left(\Delta_{gf}\right)} \det\left(C^g\right) \otimes \det(Y) \\ \det\left(\tilde{\Delta}\right)\otimes 1 \uparrow & & \uparrow 1\otimes\det\left(\Delta_f\right) \\ \left(\det\left(C^g\right) \otimes \det\left(C^f\right)\right) \otimes \det(X) & \xrightarrow{\text{ass}_{\mathcal{P}}} & \det\left(C^g\right) \otimes \left(\det\left(C^f\right) \otimes \det(X)\right) \end{array}$$

- Commutativity: Given two objects X and Y in $\mathcal{C}$, there are

$$\Delta_1 : \qquad X \xrightarrow{i_1} X \sqcup Y \xrightarrow{p_2} Y,$$

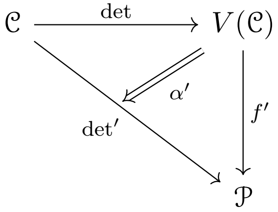
$$\Delta_2 : \qquad Y \xrightarrow{i_2} X \sqcup Y \xrightarrow{p_1} X,$$

then the following diagram commutes:

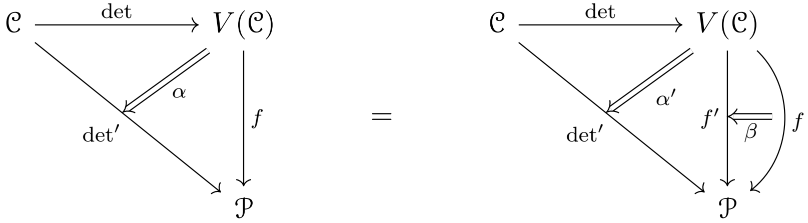
$$\begin{array}{ccc} & \det(X \sqcup Y) & \\ {\scriptstyle \det(\Delta_1)} \nearrow & & \nwarrow {\scriptstyle \det(\Delta_2)} \\ \det(Y) \otimes \det(X) & \xrightarrow[\mathrm{comm}_{\mathcal{P}}]{} & \det(X) \otimes \det(Y) \end{array}$$

$$
\begin{array}{ccccc}
f(\det\left(C^f\right)) \otimes f(\det(X)) & \xrightarrow{\mathrm{mult}_f} & f(\det\left(C^f\right) \otimes \det(X)) & \xrightarrow{f(\det(\Delta))} & f(\det(Y)) \\
\Big\downarrow{\scriptstyle \alpha\left(C^f\right)\otimes\alpha(X)} & & & & \Big\downarrow{\scriptstyle \alpha(Y)} \\
\det'\left(C^f\right) \otimes \det'(X) & & \xrightarrow[\det'(\Delta)]{} & & \det'(Y)
\end{array}
$$

Moreover, if there is another factorization

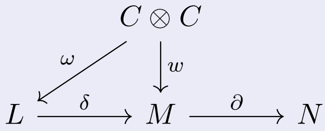


then there exists a unique tensor natural transformation $\beta : f \to f'$ such that



Definition 15

A quadratic 2-module $(\omega, \delta, \partial)$ is a diagram



1. Commutators in L satisfy the following:

$$[l, l'] = -l - l' + l + l' = \omega(\{\delta l\} \otimes \{\delta l'\}) \text{ for all } l, l' \in L.$$

Remark

Homotopy groups $\pi_n(\sigma)$ of a quadratic 2-module $\sigma = (\omega, \delta, \partial)$ are defined as follows:

- $\pi_1(\sigma) = \mathrm{coker}(\partial)$
- $\pi_2(\sigma) = \ker(\partial)/\mathrm{im}(\delta)$
- $\pi_3(\sigma) = \ker(\delta)$

An Algebraic Model for Connected 2-type

A stable quadratic module σ is a commutative diagram of group homomorphism

$$\begin{array}{ccc} N^{ab} \otimes N^{ab} & & \\ {\scriptstyle \omega}\downarrow & \searrow^{\text{commutator}} & \\ M & \xrightarrow{\partial} & N \end{array}$$

such that

1. $\omega(\{\partial m\} \otimes \{\partial m'\}) = [m, m'] = -m - m' + m + m'$, for $m, m' \in M$,
2. $\omega(\{n\} \otimes \{n'\} + \{n'\} \otimes \{n\}) = 0$, for $n, n' \in N$.

Remark

The homotopy groups of σ are:

- $\pi_0(\sigma) = \text{coker } \partial$,
- $\pi_1(\sigma) = \ker \partial$.

Suspension

Smash product

Let X, Y be two spaces. Then their smash product $X \wedge Y := X \times Y / X \vee Y$.

Example 4

$S^1 \wedge S^1 = S^2$, in fact $S^n \wedge S^m = S^{n+m}$ for any $n, m \in \mathbb{N}$.

Remark

- $\Sigma X \cong S^1 \wedge X$.
- $\Sigma^k X \cong S^k \wedge X$.

The face and degeneracy maps are defined as follows.

$$d_i(x) = (d_i^v x_0, d_{i-1}^v x_1, \cdots, d_1^v x_{i-1}, d_i^h x_{i+1}, d_i^h x_{i+2}, \cdots, d_i^h x_n),$$

$$s_i(x) = (s_i^v x_0, s_{i-1}^v x_1, \cdots, s_0^v x_i, s_i^h x_i, s_i^h x_{i+1}, \cdots, s_i^h x_n).$$

Remark

- In a category of R-modules, we have

$$\text{Hom}(X \otimes A, Y) \cong \text{Hom}(X, \text{Hom(A,Y)}).$$

- Similarly, in case of pointed topological spaces, smash product plays the role of the tensor product. If A, X are compact Hausdorff then we have

$$\text{Hom}(X \wedge A, Y) \cong \text{Hom}(X, \text{Hom(A,Y)}).$$

- So, in particular, for $A = S^1$, we have

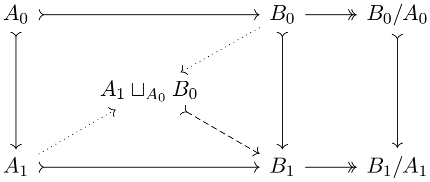
$$\text{Hom}(\Sigma X, Y) \cong \text{Hom}(X, \text{Hom}(S^1, Y)) = \text{Hom}(X, \Omega Y).$$

- Here ΩY carries compact-open topology.
- This implies, the suspension functor $\Omega \vdash \Sigma$, the loop space functor.

- Given a Waldhausen category $\mathcal{C}$, $S_2\mathcal{C}$ is also a Waldhausen category. A morphism

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

is called a cofibration if the vertical maps are cofibrations and the map from $A_1 \sqcup_{A_0} B_0 \to B_1$ is a cofibration.



- With a similar southern arrow condition, $S_n\mathcal{C}$ is a Waldhausen category for every $n \in \mathbb{N}$. (See [GH99],[Zak14].)
- Hence, one can consider $S_\bullet(S_\bullet\mathcal{C})$ and keep on doing this. This will give us a spectrum.

Definition 5

Let X and Y be two topological spaces, and let $C(X,Y)$ denote the set of all continuous maps from X to Y. Given a compact subset K of X and an open subset U of Y, let $V(K,U)$ denote the set of all functions $f \in C(X,Y)$ such that $f(K) \subseteq U$. Then the collection of all such $V(K,U)$ is a subbase for the compact-open topology on $C(X,Y)$.

- So for each bonding map $\sigma_n \colon \Sigma X_n \to X_{n+1}$, there is a corresponding map $\tilde{\sigma_n} \colon X_n \to \Omega X_{n+1}$.
- When $\tilde{\sigma}_n$ is a weak equivalence for each n, X is called an Ω-spectrum.

Detailed SQuad structure for a Waldhausen category[6]

- The generators for dimension 0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for dimension 1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- such that the following relations hold (i.e., we define ∂, w):
  - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
  - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - $[0] = 0$.
  - $[A \xrightarrow{id} A] = 0$.
  - $[A \xrightarrow{id} A \twoheadrightarrow 0] = 0, [0 \rightarrowtail A \xrightarrow{id} A] = 0$.
  - For any composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

[6] Fernando Muro and Andrew Tonks. “The 1-type of a Waldhausen K-theory spectrum”. In: *Advances in Mathematics 216* (2007), pp. 179–183.

- For any $A, B \in Ob(\mathcal{C})$, define the w as follows:

$$w([A] \otimes [B]) := \langle [A], [B] \rangle$$
$$=$$
$$-[\ B \overset{i_2}{\rightarrowtail} A \coprod B \overset{p_1}{\twoheadrightarrow} A\ ] + [\ A \overset{i_1}{\rightarrowtail} A \coprod B \overset{p_2}{\twoheadrightarrow} B\ ].$$

Here,
$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \coprod B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$
are natural inclusions and projections of a coproduct in $\mathcal{C}$.

- For any commutative diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

we have

$$[A_0 \xrightarrow{\sim} A_1] + [B_0/A_0 \xrightarrow{\sim} B_1/A_1] + \langle [A], -[B_1/A_1] + [B_0/A_0] \rangle$$
$$=$$
$$-[A_1 \rightarrowtail B_1 \twoheadrightarrow B_1/A_1] + [B_0 \xrightarrow{\sim} B_1] + [A_0 \rightarrowtail B_0 \twoheadrightarrow B_0/A_0].$$

- For any commutative diagram consisting of cofiber sequences in $\mathcal{C}$ as follows:

$$
\begin{array}{ccccc}
 & & & & C/B \\
 & & & & \uparrow\!\!\!\!\twoheadrightarrow \\
 & & B/A & \rightarrowtail & C/A \\
 & & \uparrow & & \uparrow \\
A & \rightarrowtail & B & \rightarrowtail & C
\end{array}
$$

we have,

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

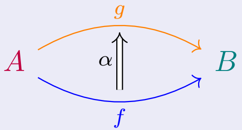
$$=$$

$$[\mathrm{A} \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle.$$

Definition 6

A (strict) 2-category $\mathcal{C}$ is comprised of the following:

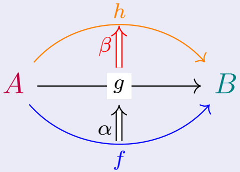
- 0-Cells (Objects): Denoted by $Ob(\mathcal{C})$.
- 1-Cells (Morphisms): For $A, B \in Ob(\mathcal{C})$, a set $\mathrm{Hom}(A, B)$ of 1-cells from A to B, also known as morphisms. A 1-cell is often written textually as $f : A \to B$ or graphically as $A \xrightarrow{f} B$.
- 2-Cells: For $A, B \in Ob(\mathcal{C}), f, g \in \mathrm{Hom}(A, B)$, a set $\mathrm{Face}(f, g)$ of 2-cells from f to g. A 2-cell is often written textually as $\alpha : f \Rightarrow g : A \to B$ or graphically as follows:



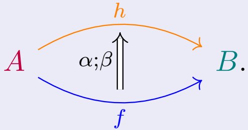
- 1-Composition: For each chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, a 1-cell $A \xrightarrow{f;g} C$.

Definition 6

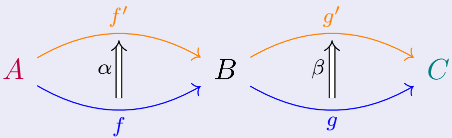
- Vertical 2-Composition: For a chain of 2-cells



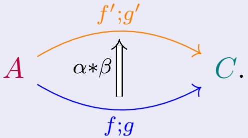
, a 2-cell



- Horizontal 2-Composition: For each chain of 2-cells

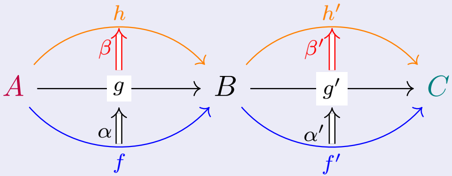


, a 2-cell



Definition 6

- Associativity: For all the compositions.
- Identities of 1-cells and 2-cells exist and are compatible with all the compositions.
- 2-Interchange: Every clover of 2-cells



satisfies

$(\alpha;\beta) * (\alpha';\beta') = (\alpha * \alpha');(\beta * \beta')$.

Definition 7

Let $\mathcal{C}$ be a Waldhausen category and $\mathbb{P}$ be a Picard 2-groupoid. A 2-determinant functor $\mathbb{D}\mathrm{et} : \mathcal{C} \to \mathbb{P}$ consists of a functor, To apdx

Definition 7

Let $\mathcal{C}$ be a Waldhausen category and $\mathbb{P}$ be a Picard 2-groupoid. A 2-determinant functor $\mathbb{D}\mathrm{et} : \mathcal{C} \to \mathbb{P}$ consists of a functor, To apdx

$$\mathbb{D}\mathrm{et} : \mathrm{we}(\mathcal{C}) \to \mathbb{P},$$

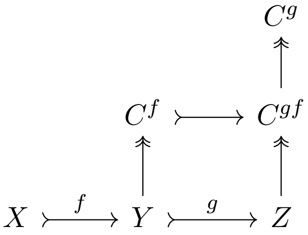
with additive data: for any Δ, a morphism

$$\det(\Delta)\colon \det\left(C^f\right) \otimes \det(X) \to \det(Y)$$

natural with respect to weak equivalences, and the following conditions must be satisfied.

- Associativity: Given a staircase

$\Theta_{f,g}$:



- Associativity: Given a staircase

$$\Theta_{f,g}: \quad \begin{array}{ccccc} & & & & C^{g} \\ & & & & \uparrow \\ & & C^{f} & \rightarrowtail & C^{gf} \\ & & \uparrow & & \uparrow \\ X & \overset{f}{\rightarrowtail} & Y & \overset{g}{\rightarrowtail} & Z \end{array}$$

The following 2-morphism exists in $\mathbb{P}$:

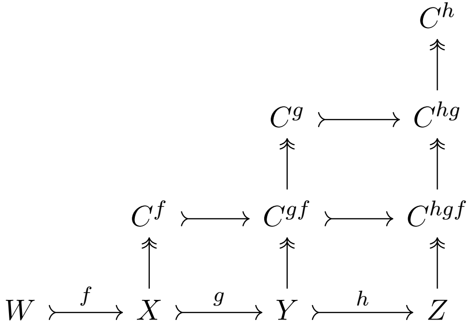
$$\begin{array}{ccc} & \mathbb{D}\mathrm{et}(Z) & \\ \overset{\mathbb{D}\mathrm{et}(\Delta_g)}{\nearrow} & & \overset{\mathbb{D}\mathrm{et}(\Delta_{gf})}{\nwarrow} \\ \mathbb{D}\mathrm{et}\left(C^{gf}\right) \otimes \mathbb{D}\mathrm{et}(X) & \overset{\mathbb{D}\mathrm{et}(\Theta_{f,g})}{\Longleftarrow} & \mathbb{D}\mathrm{et}(C^{g}) \otimes \mathbb{D}\mathrm{et}(Y) \\ \Big\uparrow{\scriptstyle \mathbb{D}\mathrm{et}(\tilde{\Delta})\otimes 1} & & \Big\uparrow{\scriptstyle 1\otimes \mathbb{D}\mathrm{et}(\Delta_f)} \\ (\mathbb{D}\mathrm{et}(C^{g}) \otimes \mathbb{D}\mathrm{et}(C^{f})) \otimes \mathbb{D}\mathrm{et}(X) & \xrightarrow[ass_{\mathbb{P}}]{} & \mathbb{D}\mathrm{et}(C^{g}) \otimes (\mathbb{D}\mathrm{et}(C^{f}) \otimes \mathbb{D}\mathrm{et}(X)) \end{array}$$

such that

Given a staircase as follows:

$$
\begin{array}{ccccccc}
 & & & & & & C^h \\
 & & & & & & \uparrow\!\!\uparrow \\
 & & & & C^g & \rightarrowtail & C^{hg} \\
 & & & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
 & & C^f & \rightarrowtail & C^{gf} & \rightarrowtail & C^{hgf} \\
 & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\
W & \overset{f}{\rightarrowtail} & X & \overset{g}{\rightarrowtail} & Y & \overset{h}{\rightarrowtail} & Z
\end{array}
$$

Given a staircase as follows:



We have the following cocycle condition:

$$\mathbb{D}\text{et}\left(\Theta_{f,hg}\right) \cdot \mathbb{D}\text{et}\left(\Theta_{g,h}\right) = \left(\mathbb{D}\text{et}\left(\Theta_{g,h}\right) \otimes \mathbb{D}\text{et}(W)\right) \cdot \mathbb{D}\text{et}\left(\Theta_{gf,h}\right) \cdot \left(\mathbb{D}\text{et}\left(C^h\right) \otimes \mathbb{D}\text{et}\left(\Theta_{f,g}\right)\right). \quad (1)$$

- Commutativity: Given two objects X and Y in $\mathcal{C}$, the following 2-morphism exists in $\mathbb{P}$:

$$\begin{array}{ccc} & \mathbb{D}\mathrm{et}(X \sqcup Y) & \\ \nearrow^{\mathbb{D}\mathrm{et}(\Delta_2)} & \Leftarrow_{\eta_{X,Y}} & \nwarrow^{\mathbb{D}\mathrm{et}(\Delta_1)} \\ \mathbb{D}\mathrm{et}(X) \otimes \mathbb{D}\mathrm{et}(Y) & \xrightarrow[\mathrm{comm}_{\mathcal{P}}]{} & \mathbb{D}\mathrm{et}(Y) \otimes \mathbb{D}\mathrm{et}(X) \end{array}$$

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such that, given three objects X, Y, Z in $\mathcal{C}$,

$$\Theta_{X,Y,Z}\colon \quad \begin{array}{ccccc} & & & & Z \\ & & & & \uparrow\!\!\uparrow \\ & & Y & \rightarrowtail & Y \sqcup Z \\ & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ X & \rightarrowtail & X \sqcup Y & \rightarrowtail & X \sqcup Y \sqcup Z \end{array}$$

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such that, given three objects X, Y, Z in $\mathcal{C}$,

$$\Theta_{X,Y,Z} : \quad \begin{array}{ccccc} & & & & Z \\ & & & & \uparrow\!\!\uparrow \\ & & Y & \rightarrowtail & Y \sqcup Z \\ & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ X & \rightarrowtail & X \sqcup Y & \rightarrowtail & X \sqcup Y \sqcup Z \end{array}$$

Then the following holds.

$$(\mathrm{LHex}_{\mathbb{P}})_{\mathbb{D}\mathrm{et}(X)|\mathbb{D}\mathrm{et}(Y),\mathbb{D}\mathrm{et}(Z)} \cdot (\mathbb{D}\mathrm{et}(Y) \otimes \eta_{X,Z}) \cdot \mathbb{D}\mathrm{et}\left(\Theta_{Z,X,Y}\right) \cdot (\eta_{X,Y} \otimes \mathbb{D}\mathrm{et}(Z))$$
$$= \mathrm{Det}(\Theta_{X,Z,Y}) \cdot \eta_{X,Y \coprod Z} \cdot \mathbb{D}\mathrm{et}\left(\Theta_{Z,Y,X}\right).$$

Definition 8

We define the 2-groupoid of virtual objects in the same way as we did in definition **??**.

Proposition 9

Let $\mathcal{C}$ be a Waldhausen category. The naturality of the monoidal product in the Picard 2-groupoid of the virtual objects of $\mathcal{C}$ is tied to the ω of the quadratic module associated to $\mathcal{C}$.

Proof.

Let $\mu_{f,g}$ be the 2-morphism that exists due to the naturality of the braiding as follows.

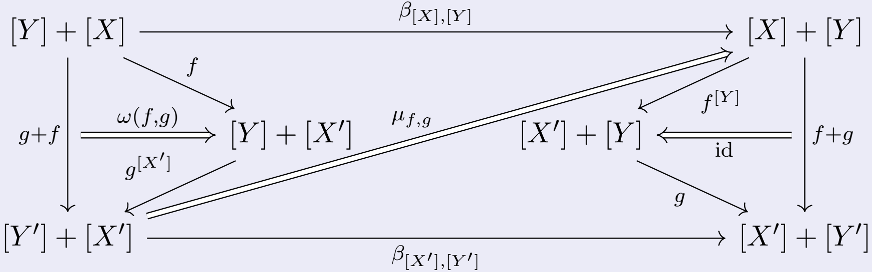
$$\begin{array}{ccc} [Y]+[X] & \xrightarrow{\beta_{[X],[Y]}} & [X]+[Y] \\ {\scriptstyle g+f}\downarrow & \overset{\mu_{f,g}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle f+g} \\ [Y']+[X'] & \xrightarrow[\beta_{[X'],[Y']}]{} & [X']+[Y'] \end{array}$$

This translates to the following equation.

$$\delta\left(\mu_{f,g}\right) = -\beta_{[X'],[Y']} - f - g^{[X]} + \beta_{[X],[Y]} + f^{[Y]} + g.$$

Proof.

This expands to give us the following diagram.



Here $\omega(f,g) = \omega\left(\{[f]\} \otimes \left\{\left[g^{[X]}\right]\right\}\right)$.
This can also be viewed as the monoidal product $+ \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ being an oplax functor. □

Oplax functor

If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that, for 1-cells f, g, we have $F(f \circ g) \cong F(f) \circ F(g)$ (but not exactly equal). Then the functor F is called as an oplax functor.

$$H^2(B,A) \underset{\psi}{\overset{\varphi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} CentrExt(B,A)$$

Proof.

For $[c] \in H^2(B,A)$: $c : B \times B \to A$. Define a group $E = U(G) \times U(A)$ with,

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 \cdot b_2, a_1 + a_2 + c(b_1, b_2)).$$

The map $E \xrightarrow{p} B$ is projection, and $A \xrightarrow{i} E$ is inclusion.
For the other direction, choose a section $s : U(B) \to U(E)$, define $c : B \times B \to A$ as

$$c(b_1, b_2) = s(b_1)s(b_2)s(b_1 b_2)^{-1}.$$

Applying p to $c(b_1, b_2)$ gives us identity, hence $c(b_1, b_2) \in \ker(p)$, hence by exactness, it belongs to A. □

- Let G be a group, S be a right G-torsor and T be a left G-torsor, then we define the contracted product of S and T by the G-torsor:

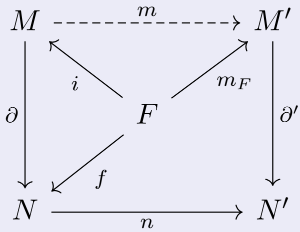
$$S \wedge^A T := S \times T/[(sg, t) \sim (s, gt)].$$

where G acts on $[(s, t)]$ as

$$g \cdot [(s, t)] = [(sg, t)].$$

Definition 10

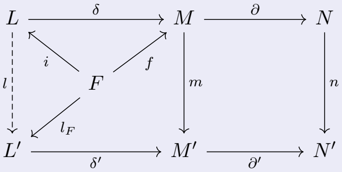
Let F be a free group and let $f\colon F \to N$ be a homomorphism. We say that a nil(n)-module $\partial\colon M \to N$ is a free nil(n)-module with basis f, if a homomorphism $i\colon F \to M$ is given such that $\partial \circ i = f$ and such that the following universal property is satisfied. For each nil(n)-module $\partial'\colon M' \to N'$ and for each commutative diagram of unbroken arrows



there is a unique map $(m, n)\colon \partial \to \partial'$ in cross(n)—the category of nil(n)-modules. A nil(n)-module $\partial\colon M \to N$ is called totally free if ∂ is free as above and N is a free group.

Definition 11

Let $\partial\colon M \to N$ be a nil(2)-module and let $f\colon F \to M$ be a homomorphism where F is a free group and for which $\partial \circ f = 0$. We say that a quadratic module $(\omega, \delta, \partial)$ is a free quadratic module with basis f if a homomorphism $i\colon F \to L$ is given such that $\delta \circ i = f$ and the following universal property is satisfied. Consider any commutative diagram of broken arrows



where $(\omega', \delta', \partial')$ is a quadratic module, where $(m, n)\colon \partial \to \partial'$ is a map of nil(2)-modules, l_F is a homomorphism, then there exists a unique map (l, m, n) in the category of quadratic modules. We say $(\omega, \delta, \partial)$ is a totally free quadratic module if it is free and ∂ is a totally free nil(2)-module.

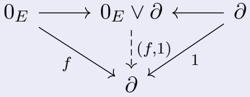
Construction—Free Nil(n)-module

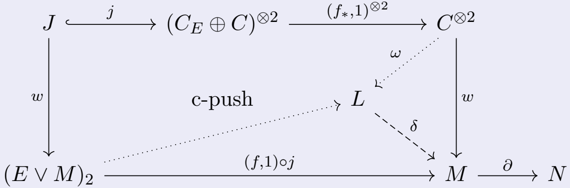
A free(n)-module with basis $f\colon \langle Z\rangle \to N$ can be constructed as follows. Define a pre-crossed module $\partial_f\colon \langle Z \times N\rangle \to N$ where $\partial_f(x,\alpha) = -\alpha + f(x) + \alpha$, and the action is defined as $(x,\alpha)^\beta = (x, \alpha+\beta)$. Then a free nil($n$)-module can be constructed by modding out by the $(n+1)^{\text{th}}$ Peiffer commutator.

$$r_n(\partial_f)\colon \langle Z \times N\rangle / P_{n+1}(\partial_f) \to N$$

Construction—Free Quadratic Module

We can now construct a free quadratic module from a given nil(2)-module $\partial\colon M \to N$ and a basis $f\colon F \to L$. Using universal property of sum of nil(n)-modules, we have





where $C = (M^{\text{cr}})^{\text{ab}}$, C_E is the free $(\pi_1\partial)$-module generated by Z , $(E \vee M)_2 = \ker(0,1)$, $J = (C_E \otimes C_E) \oplus (C_E \otimes C) \oplus (C \otimes C_E)$, and w is the Peiffer commutator. The pushout square is called central because the vertical maps need to be central.

Definition 12

A (p, q)-*shuffle* is a pair of strictly increasing functions

$$\sigma_0 \colon \{0, 1, \cdots, q-1\} \longrightarrow \{0, 1, \cdots, p+q-1\}$$

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with disjoint images.

Each shuffle (σ_0, σ_1) has a sign, $\mathrm{sg}(\sigma)$, where σ is the permutation of $\{0, \cdots, p+q-1\}$ as follows

$$\sigma(i) = \begin{cases} \sigma_1(i) & \text{for } i < p \\ \sigma_0(i-p) & \text{for } p \leq i < p+q \end{cases}$$

Proposition 13

Let K, L be reduced simplicial sets. Let $\rho(K), \rho(L)$ denote the associated crossed complexes. There exists a crossed complex homomorphism

$$b\colon \rho(K) \otimes \rho(L) \longrightarrow \rho(K \times L)$$

natural in K, L, defined for $x \in K_p, y \in L_q$ by

$$b(x \otimes y) = \begin{cases} -(s_0 x, s_1 y) + (s_1 x, s_0 y) & \textit{for } (p,q) = (1,1) \\ \displaystyle\sum_{(\sigma_0,\sigma_1)\in S_{p,q}} sg(\sigma)\cdot(s_{\sigma_0} x, s_{\sigma_1} y) & \textit{for } (p,q) \neq (1,1) \end{cases}$$

Definition 14

An $S_\bullet$-category $\mathcal{C}_\bullet$ is a simplicial category such that

- $\mathcal{C}_0 = *$, $\mathcal{C}_n$ has finite coproducts for all $n \geq 0$,
- Faces and degeneracies preserve the coproducts.

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Example 15

Given a Waldhausen category $\mathcal{C}$, $S_\bullet\mathcal{C}$ is an $S_\bullet$-category.

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Remark

- For an $S_\bullet$-category $\mathcal{C}_\bullet$, and a Picard 2-groupoid $\mathbb{P}$, one can define 2-determinant functors $\mathbb{D}\mathrm{et}\colon \mathcal{C}_\bullet \to \mathbb{P}$ (see [MTW15]).
- One can also define a quadratic module and the associated Picard 2-groupoid that models the 2-groupoid of virtual objects of $\mathcal{C}_\bullet$.

- A biextension $E \to B \times B$ is *anti-symmetric* if the symmetric biextension $E \wedge \sigma^* E$ is trivial as a symmetric biextension,
  - $\sigma \colon B \times B \to B \times B$ is the permutation map.
- E is *alternating* if $\Delta^* E$ itself is split, in a manner compatible with the canonical splitting of $(\Delta^* E)^2$.
  - $\Delta \colon B \to B \times B$ is the diagonal
- The monoidal structure on $\mathcal{E}$ is symmetric precisely when the associated biextension given by the commutator map is alternating *and* the underlying biextension admits a trivialization compatible with the anti-symmetric structure.