Stabilization of 2-Crossed Modules

Milind Gunjal
In collaboration with Dr. Ettore Aldrovandi

Department of Mathematics
Florida State University

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Gadget

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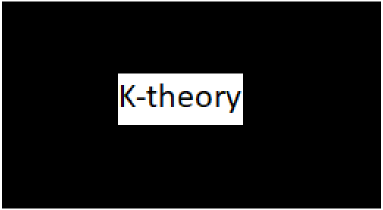

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Space with interesting homotopy groups

- Examples of such gadgets.
  - Category of finitely generated projective R-modules.
    If X is the output of its K-theory, then we have:
    - $\pi_1(X) = K_0(R)$.
    - $\pi_2(X) = K_1(R) = R^\times =$ Units of R.
  - A Waldhausen Category.

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    - ★ $\pi_2(X) = K_1(R) = R^\times =$ Units of R.
  - A Waldhausen Category.

Definition 1

A Waldhausen category[a] $\mathcal{C}$ is a category with a zero object, 0 equipped with two classes of morphisms: weak equivalences (WE) and cofibrations (CO) such that it has a notion of taking quotients, and satisfy certain conditions.

[a] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory.* American Mathematical Society, 2010, pp. 172–174.

Examples of Waldhausen categories

1. The category **R-Mod**, for any ring R.
   - Injective maps (CO).
   - Isomorphisms (WE).
2. An exact category.
   - Monomorphisms (CO).
   - Isomorphisms (WE).

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    - Injective maps (CO).
    - Isomorphisms (WE).
2. An exact category.
    - Monomorphisms (CO).
    - Isomorphisms (WE).
3. Category $\mathcal{R}(X)$ of spaces that retract to X.
    - Serre cofibrations (CO).
    - Maps that induce isomorphisms for chosen homology theory (WE).
4. The category of finite sets.
    - Inclusions (CO).
    - Isomorphisms (WE).

Gadget

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K-theory

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Space with interesting homotopy groups

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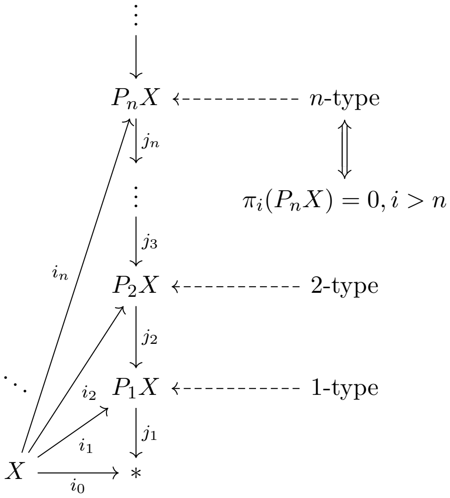
n-types

n-types


P_nX
n-type
j_n
$\pi_i(P_nX) = 0, i > n$
j_3
i_n
P_2X
2-type
j_2
i_2
P_1X
1-type
j_1
i_1
X
$*$
i_0

Algebraic model of a 1-type

Groups can be considered as algebraic models for the 1-type.

- If a space X is such that,

$$\pi_i(X) = \begin{cases} G & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}$$

- $BG := |N(G \rightrightarrows *)|$.
- $X \simeq BG$.

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- $X \simeq BG$.

n-types	Categorical model	Algebraic model
1-type	$\mathcal{G} = (G \rightrightarrows *)$	G

Theorem 2 (Homotopy Hypothesis (Grothendieck))

By taking classifying spaces and fundamental n-groupoids, there is an equivalence between the theory of weak n-goupoids and that of homotopy n-types.

n-types	Categorical model	Algebraic model	Groups
0-type	0-category	Set	
1-type	1-category	Group	1 group
2-type	2-category	Crossed Module[1]	2 groups
3-type	3-category	2-Crossed Module	3 groups

[1] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Definition 3

A 2-crossed module[a] G_* consists of a complex of G_0-groups

$$\begin{array}{ccccc} G_1 \times G_1 & & & & \\ \big\downarrow \{\cdot,\cdot\} & & & & \\ G_2 & \xrightarrow{\partial_2} & G_1 & \xrightarrow{\partial_1} & G_0 \end{array}$$

- ∂'s are G_0-equivariant.
- $G_2 \xrightarrow{\partial_2} G_1$ is a crossed module.
  - ∂_2 is G_1-equivariant.
  - $f^{\partial_2 g} = g^{-1} f g$ for all $f, g \in G_2$.

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  - ∂_2 is G_1-equivariant.
  - $f^{\partial_2 g} = g^{-1} f g$ for all $f, g \in G_2$.
- $(\alpha^f)^x = (\alpha^x)^{f^x}$ for all $\alpha \in G_2, f \in G_1, x \in G_0$.
- Compatibility conditions.

[a] Ronald Brown and İlhan İçen. "Homotopies and Automorphisms of Crossed Modules of Groupoids". In: *Applied Categorical Structures* (2003), p. 193.

Remark

The homotopy groups of a 2-crossed module G_* are:

- $\pi_0(G_*) = \mathrm{Coker}(\partial_1 : G_1 \to G_0)$,
- $\pi_1(G_*) = \mathrm{Ker}(\partial_1 : G_1 \to G_0)/(\mathrm{Im}(\partial_2 : G_2 \to G_1))$,
- $\pi_2(G_*) = \mathrm{Ker}(\partial_2 : G_2 \to G_1)$.

Current work

- From a given Waldhausen category, it is known that we can get a group (1-type), and a stable crossed module (2-type)[2].
- Now, we want to find a 3-type using the same procedure to get the 2-crossed module G_*.

[2]Fernando Muro and Andrew Tonks. “The 1-type of a Waldhausen K-theory spectrum”. In: *Advances in Mathematics 216* (2007), pp. 179–183.

- The generators for G_0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for G_1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any WE.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.

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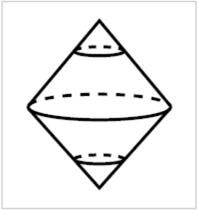
- The generators for G_2 are:
  - $$\begin{array}{ccccc} & & A_2 & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ A_0 & & \xrightarrow{\quad\sim\quad} & & A_1 \end{array}$$
  - $$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$
  - $$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow\!\!\uparrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$
- But this is not stable yet. So we make it stable by realizing the monoidal 2-Cat structure on it.

Stability

- The output of K-theory is in fact a spectrum $\mathbb{X}$, i.e., a sequence of pointed spaces $\{X_n\}_{n\geq 0}$ with the structure maps $\Sigma X_n \to X_{n+1}$.

Definition 4

For a space X, the suspension ΣX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. ($\Sigma X = S^1 \wedge X$).



Example: $\Sigma S^n = S^{n+1}$

Theorem 5 (Freudenthal Suspension Theorem)

For a spectrum $\mathbb{X} = \{X_n\}_{n\geq 0}$*, the sequence*

$$\pi_i(X_n) \to \pi_{i+1}(X_{n+1}) \to \pi_{i+2}(X_{n+2}) \to \cdots$$

eventually stabilizes.

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Stable Homotopy Group

The i^{th} stable homotopy group of $\mathbb{X}$ is:

$$\pi_i^s(\mathbb{X}) = \varinjlim_{k} \pi_{i+k}(X_k) \cong \pi_{i+N}(X_N), \ N \gg 0.$$

Theorem 6 (The Stable Homotopy Hypothesis)

[a] *Symmetric monoidal structure corresponds to topological stability.*

$$
\begin{array}{ccccc}
\textit{Stable 1-types} & \longleftrightarrow & \textit{Symmetric Monoidal Categories} & \longleftrightarrow & \textit{Stable Crossed Module} \\
\downarrow & & \downarrow & & \downarrow \\
\textit{Stable 2-types} & \longleftrightarrow & \textit{Symmetric Monoidal 2-Categories} & \longleftrightarrow & \textit{Stable 2-Crossed Modules} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}
$$

[a] Niles Johnson Nick Gurski and Angélica M. Osorno. "The 2-dimensional stable homotopy hypothesis". In: *Journal of Pure and Applied Algebra, Volume 223, Issue 10, 2019* (2019), pp. 4348–4383.

SM 2-Cat structure on a 2-CM

- Given a 2-CM G_*

$$G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0$$

- $Ob(\Gamma(G_*)) = G_0$.

$$x_0 \in G_0.$$

- 1-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1$.

$$x_0 \xrightarrow{f_0} x_1 \text{ such that } x_1 = x_0 \cdot \partial(f_0).$$

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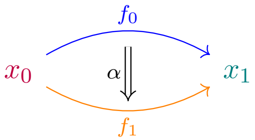
- 2-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1 \rtimes G_2$.



Such that $f_1 = f_0 \cdot \partial(\alpha)$.

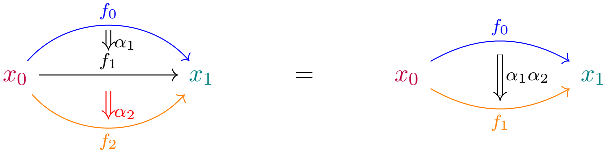
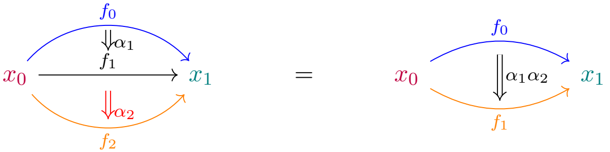


Figure 1: Vertical composition



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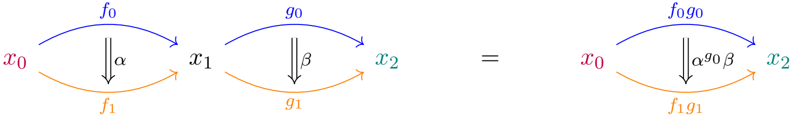
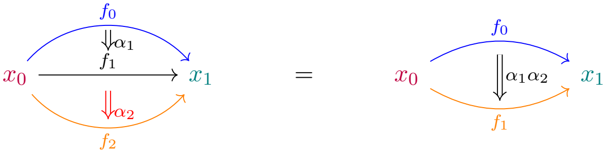


Figure 2: Horizontal composition

They satisfy certain compatibility conditions.



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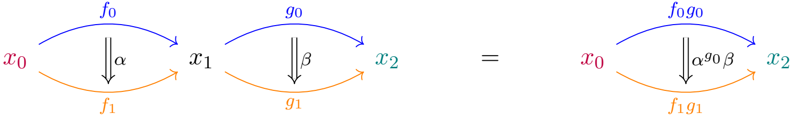

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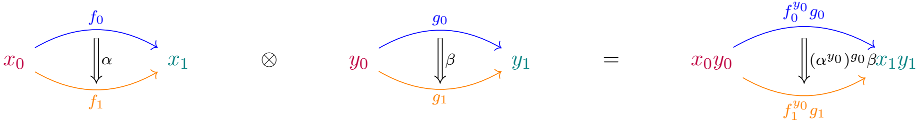

Figure 3: Monoidal structure

Components of a Symmetric Monoidal 2-Category[3] (SM 2-Cat) are:

- A 2-Cat.
- Monoidal structure ($\otimes$) on the 2-Cat.

[3] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, 2021, pp. 384–396.

Components of a Symmetric Monoidal 2-Category[3] (SM 2-Cat) are:

- A 2-Cat.
- Monoidal structure ($\otimes$) on the 2-Cat.
- Braiding (β) on the monoidal structure.
- Left ($\eta_{-|--}$) and right ($\eta_{--|-}$) hexagonators.
- Syllepsis (γ) (Exclusive for 2-Cat).
  - Symmetry axiom.

- Pull back the symmetric structure to get a stable 2-CM.

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Thank You!

References I

[1] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory*. American Mathematical Society, 2010, pp. 172–174.

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References II

[6] H.-J. Baues and Daniel Conduché. "On the 2-type of an iterated loop space". In: *Forum Mathematicum* (1997), pp. 725–733.

Waldhausen category

A Waldhausen category[a] $\mathcal{C}$ is a category with a zero object, 0 equipped with two classes of morphisms: weak equivalences (WE) and cofibrations (CO) such that it has a notion of taking quotients, and satisfy certain conditions.

- $\text{iso}(\mathcal{C}) \subseteq \text{WE}(\mathcal{C}) \cap \text{CO}(\mathcal{C})$.
- $0 \to X \in \text{CO}(\mathcal{C})$ for all $X \in \text{Ob}(\mathcal{C})$.
- If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \bigcup_A C$ of these two maps exists in $\mathcal{C}$ and $C \rightarrowtail B \bigcup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \bigcup_A C \end{array}$$

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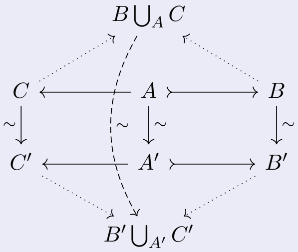
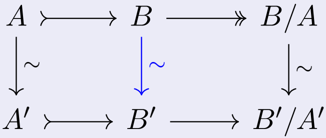
Waldhausen category

- Gluing axiom:



The induced map $B \bigcup_A C \to B' \bigcup_{A'} C'$ is also a weak equivalence.

- Extension axiom:



If $A \to A'$ and $B/A \to B'/A'$ are w.e. then so is $B \to B'$.

Serre cofibrations

- In the category of topological spaces, a map $f : X \to Y$ is called a Serre fibration, if for each CW-complex A, the map f has the RLP w.r.t. the inclusion $A \times \{0\} \to A \times [0,1]$:

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow f \\ A \times [0,1] & \longrightarrow & Y \end{array}$$

- A map f is called a Serre cofibration if it has the LLP w.r.t. acyclic fibrations.

Crossed Module

Definition 7

A crossed module[a] G_* consists of a G_0-equivariant group homomorphism, where G_0 acts on itself by conjugation.

$$G_1 \xrightarrow{\partial} G_0$$

where the action of G_0 on G_1 satisfies

- $f^{\partial g} = g^{-1} f g$ for all $f, g \in G_1$.

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Remark

The homotopy groups of the crossed module G_* are:

- $\pi_0(G_*) = \text{Coker } \partial$,
- $\pi_1(G_*) = \text{Ker } \partial$.

Extending the previous idea for higher values of n:

$$X \simeq |N\mathcal{G}| \tag{1}$$

- $n = 2$. For a given crossed module G_*, we can construct a category $\Gamma(G_*)$ such that
  - Ob $(\Gamma(G_*)) = G_0$
  - 1-Mor $(\Gamma(G_*)) = G_0 \rtimes G_1$
    - G_1 acts on G_0 by sending $x_0 \mapsto x_0 \cdot \partial f$ for $f \in G_1$.
- For equation 1, $\mathcal{G} = (\Gamma(G_*) \rightrightarrows *)$ works.

Stable Crossed Module

Definition 8

A stable crossed module (SCM)[a] G_* is a crossed module $\partial : G_1 \to G_0$ together with a map Back to main

$$\langle \cdot, \cdot \rangle : G_0 \times G_0 \to G_1$$

satisfying the following for any $f, g \in G_1, x, y, z \in G_0$:

1. $\partial\langle x, y\rangle = [y, x]$,
2. $f^x = f + \langle x, \partial(f)\rangle$,
3. $\langle x, y + z\rangle = \langle x, y\rangle^z + \langle x, z\rangle$,
4. $\langle x, y\rangle + \langle y, x\rangle = 0$.

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Some facts

- Examples of a model category which is not a Waldhausen category: Triangulated categories.
- The functor $\_ \otimes \_ : \Gamma(G_*) \times \Gamma(G_*) \to \Gamma(G_*)$ is in fact an oplax functor.

Oplax functor

If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that, for 1-cells f, g, we have $F(f \circ g) \cong F(f) \circ F(g)$ (but not exactly equal). Then the functor F is called as an oplax functor.

Suspension

Smash product

Let X, Y be two spaces. Then their smash product
$X \wedge Y := X \times Y / X \vee Y$.

Example 9

$S^1 \wedge S^1 = S^2$, in fact $S^n \wedge S^m = S^{n+m}$ for any $n, m \in \mathbb{N}$.

Remark

- $\Sigma X \cong S^1 \wedge X$.
- $\Sigma^k X \cong S^k \wedge X$.

Remark

- In a category of R-modules, we have

$$\mathrm{Hom}(X \otimes A, Y) \cong \mathrm{Hom}(X, \mathrm{Hom(A,Y)}).$$

- Similarly, in case of pointed topological spaces, smash product plays the role of the tensor product. If A, X are compact Hausdorff then we have

$$\mathrm{Hom}(X \wedge A, Y) \cong \mathrm{Hom}(X, \mathrm{Hom(A,Y)}).$$

- So, in particular, for $A = S^1$, we have

$$\mathrm{Hom}(\Sigma X, Y) \cong \mathrm{Hom}(X, \mathrm{Hom}(S^1, Y)) = \mathrm{Hom}(X, \Omega Y).$$

- Here ΩY carries compact-open topology.
- This implies, the suspension functor $\Omega \vdash \Sigma$, the loop space functor.

Definition 10

Let X and Y be two topological spaces, and let $C(X, Y)$ denote the set of all continuous maps from X to Y. Given a compact subset K of X and an open subset U of Y, let $V(K, U)$ denote the set of all functions $f \in C(X, Y)$ such that $f(K) \subseteq U$. Then the collection of all such $V(K, U)$ is a subbase for the compact-open topology on $C(X, Y)$.

Definition 11

A stable quadratic module C_* is a commutative diagram of group homomorphisms To appendix

$$\begin{array}{ccc} C_0^{ab} \otimes C_0^{ab} & & \\ \downarrow w & \searrow^{\text{commutator}} & \\ C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

such that given $c_i, d_i \in C_i, i = 0, 1,$

1. $w(\{\partial(c_1)\} \otimes \{\partial(d_1)\}) = [d_1, c_1] = d_1^{-1} c_1^{-1} d_1 c_1,$
2. $w(\{c_0\} \otimes \{d_0\} + \{d_0\} \otimes \{c_0\}) = 0.$ (The stability condition).

$$C_0 \to C_0^{ab}$$

$$x \mapsto \{x\}$$

Remark

The homotopy groups of C_* are:

- $\pi_0(C_*) = \text{Coker}\partial,$
- $\pi_1(C_*) = \text{Ker}\partial.$

Detailed SQuad structure for a Waldhausen category[4]

- The generators for dimension 0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for dimension 1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- such that the following relations hold (i.e., we define ∂, w):
  - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
  - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - $[0] = 0$.
  - $[A \xrightarrow{id} A] = 0$.
  - $[A \xrightarrow{id} A \twoheadrightarrow 0] = 0, [0 \rightarrowtail A \xrightarrow{id} A] = 0$.
  - For any composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

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- For any $A, B \in Ob(\mathcal{C})$, define the w as follows:

$$w([A] \otimes [B]) := \langle [A], [B] \rangle$$

$$=$$

$$-[\ B \xrightarrow{i_2} A \coprod B \xrightarrow{p_1} A\ ] + [\ A \xrightarrow{i_1} A \coprod B \xrightarrow{p_2} B\ ].$$

Here,

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \coprod B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are natural inclusions and projections of a coproduct in $\mathcal{C}$.

- For any commutative diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

we have

$$[A_0 \xrightarrow{\sim} A_1] + [B_0/A_0 \xrightarrow{\sim} B_1/A_1] + \langle [A], -[B_1/A_1] + [B_0/A_0] \rangle$$

$$=$$

$$-[A_1 \rightarrowtail B_1 \twoheadrightarrow B_1/A_1] + [B_0 \xrightarrow{\sim} B_1] + [A_0 \rightarrowtail B_0 \twoheadrightarrow B_0/A_0].$$

- For any commutative diagram consisting of cofiber sequences in $\mathcal{C}$ as follows:

$$
\begin{array}{ccccc}
 & & & & C/B \\
 & & & & \uparrow \\
 & & B/A & \rightarrowtail & C/A \\
 & & \uparrow & & \uparrow \\
A & \rightarrowtail & B & \rightarrowtail & C
\end{array}
$$

we have,

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

$$=$$

$$[\mathrm{A} \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle.$$

Simplicial Set

A simplicial set $X \in \mathbf{sSet}$ is

- for each $n \in \mathbb{N}$ a set $X_n \in \mathbf{Set}$ (the set of n-simplices),
- for each injective map $\partial_i : [n1]ß[n]$ of totally ordered sets ($[n] := (0 < 1 < \cdots < n)$,
- a function $d_i : X_n \to X_{n1}$ (the i^{th} face map on n-simplices) ($n > 0$ and $0in$),
- for each surjective map $\sigma_i : [n+1] \to [n]$ of totally ordered sets,
- a function $s_i : X_n \to X_{n+1}$ (the i^{th} degeneracy map on n-simplices) ($n \geq 0$ and $0 \leq i \leq n$),
- such that these functions satisfy the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{when } i < j, \\ 1, & \text{when } i = j, j+1, \\ s_j d_{i-1}, & \text{when } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ when } i \leq j$$

The face maps, and degeneracy maps for the Nerve of a category are as follows:

- $d_i : N_k(\mathcal{C}) \to N_{k-1}(\mathcal{C})$:

$$\begin{array}{c}(A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots \to A_k) \\ \big\downarrow \\ (A_1 \to \cdots A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots A_k)\end{array}$$

- $s_i : N_k(\mathcal{C}) \to N_{k+1}(\mathcal{C})$:

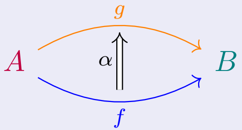
$$(A_1 \to \cdots \to A_i \to \cdots \to A_k) \mapsto (A_1 \to \cdots A_i \xrightarrow{\text{id}} A_i \to \cdots A_k).$$

2-Categories

Definition 12

A (strict) 2-category $\mathcal{C}$ is comprised of the following:

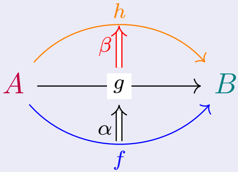
- 0-Cells (Objects): Denoted by $Ob(\mathcal{C})$.
- 1-Cells (Morphisms): For $A, B \in Ob(\mathcal{C})$, a set $\text{Hom}(A, B)$ of 1-cells from A to B, also known as morphisms. A 1-cell is often written textually as $f : A \to B$ or graphically as $A \xrightarrow{f} B$.
- 2-Cells: For $A, B \in Ob(\mathcal{C})$, $f, g \in \text{Hom}(A, B)$, a set $\text{Face}(f, g)$ of 2-cells from f to g. A 2-cell is often written textually as $\alpha : f \Rightarrow g : A \to B$ or graphically as follows:



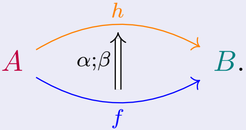
- 1-Composition: For each chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, a 1-cell $A \xrightarrow{f;g} C$.

Definition 12

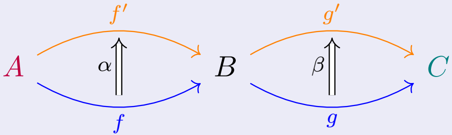
- Vertical 2-Composition: For a chain of 2-cells



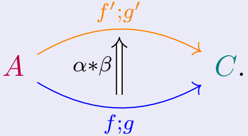
, a 2-cell



- Horizontal 2-Composition: For each chain of 2-cells

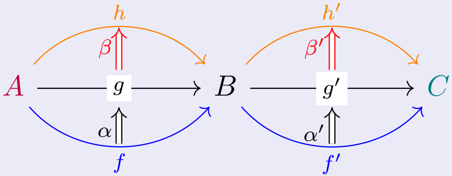


, a 2-cell



Definition 12

- Associativity: For all the compositions.
- Identities of 1-cells and 2-cells exist and are compatible with all the compositions.
- 2-Interchange: Every clover of 2-cells



satisfies

$(\alpha;\beta) * (\alpha';\beta') = (\alpha * \alpha');(\beta * \beta')$.