Group Cohomology with Values in a Picard Category

Milind Gunjal

Department of Mathematics
Florida State University

December 7$^{\text{th}}$, 2023

Definition 1

A 2-crossed module[a] G_* consists of a complex of G_0-groups

$$\begin{array}{ccccc} G_1 \times G_1 & & & & \\ \Big\downarrow{\scriptstyle \{\cdot,\cdot\}} & & & & \\ G_2 & \xrightarrow{\partial_2} & G_1 & \xrightarrow{\partial_1} & G_0 \end{array}$$

- ∂'s are G_0-equivariant.
- $G_2 \xrightarrow{\partial_2} G_1$ is a crossed module.
  - ∂_2 is G_1-equivariant.
  - $f^{\partial_2 g} = g^{-1} f g$ for all $f, g \in G_2$.

Definition 1

A 2-crossed module[a] G_* consists of a complex of G_0-groups

$$\begin{array}{c} G_1 \times G_1 \\ \downarrow \{\cdot,\cdot\} \\ G_2 \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \end{array}$$

- ∂'s are G_0-equivariant.
- $G_2 \xrightarrow{\partial_2} G_1$ is a crossed module.
  - ∂_2 is G_1-equivariant.
  - $f^{\partial_2 g} = g^{-1} f g$ for all $f, g \in G_2$.
- $(\alpha^f)^x = (\alpha^x)^{f^x}$ for all $\alpha \in G_2, f \in G_1, x \in G_0$.
- Compatibility conditions.

[a] Ronald Brown and İlhan İçen. "Homotopies and Automorphisms of Crossed Modules of Groupoids". In: *Applied Categorical Structures* (2003), p. 193.

SM 2-Cat structure on a 2-CM

- Given a 2-CM G_*

$$G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0$$

- $Ob(\Gamma(G_*)) = G_0$.

$$x_0 \in G_0.$$

- 1-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1$.

$$x_0 \xrightarrow{f_0} x_1 \text{ such that } x_1 = x_0 \cdot \partial(f_0).$$

SM 2-Cat structure on a 2-CM

- Given a 2-CM G_*

$$G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0$$

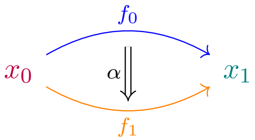
- $Ob(\Gamma(G_*)) = G_0$.

$$x_0 \in G_0.$$

- 1-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1$.

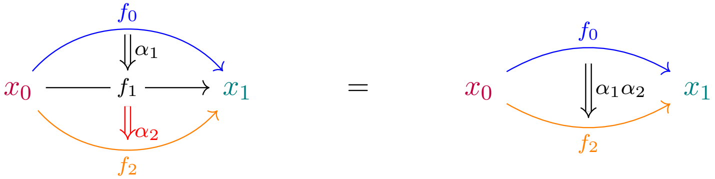
$$x_0 \xrightarrow{f_0} x_1 \text{ such that } x_1 = x_0 \cdot \partial(f_0).$$

- 2-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1 \rtimes G_2$.

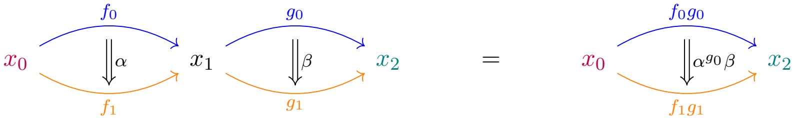


Such that $f_1 = f_0 \cdot \partial(\alpha)$.

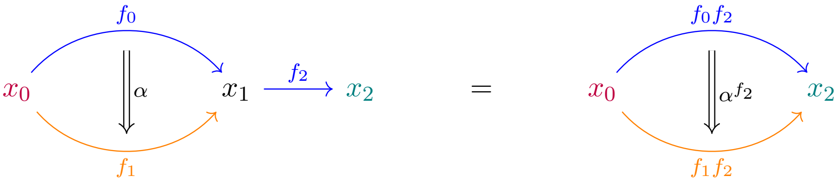
- Vertical composition

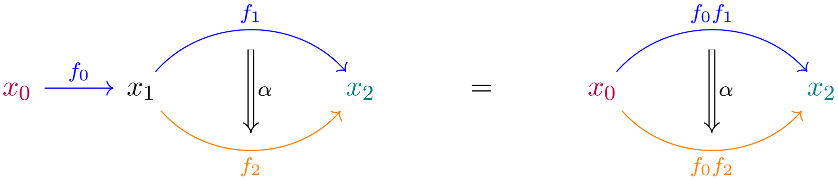


- Horizontal composition

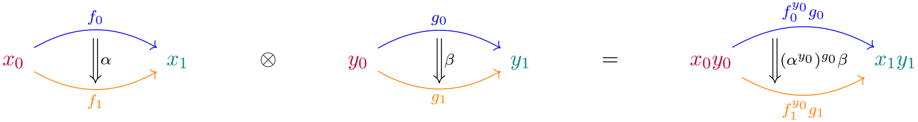


Special cases: Whiskering.

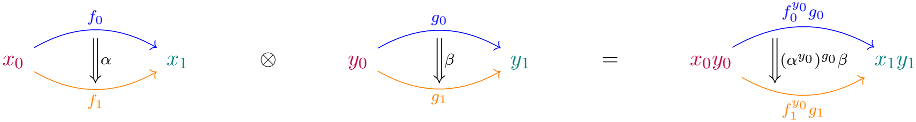
- 

- 

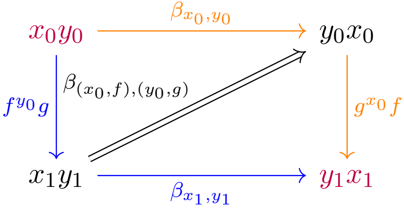
- Monoidal product



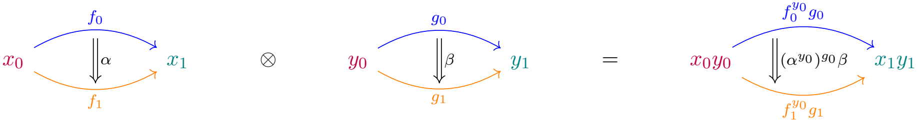
- Monoidal product



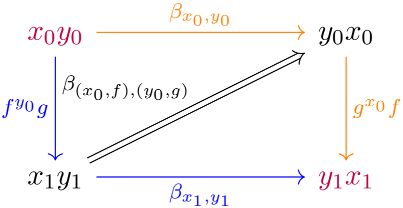
- Braiding



- Monoidal product

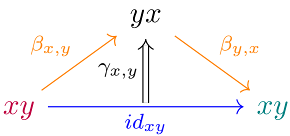


- Braiding

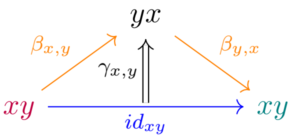


- Hexagonators

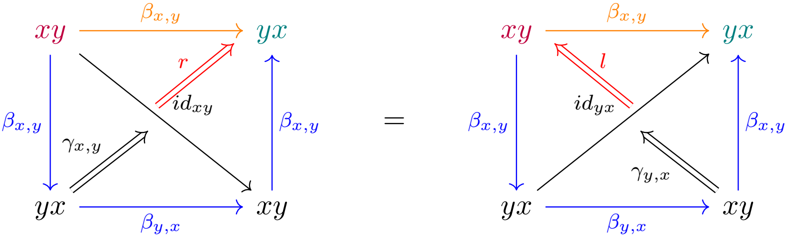
- Syllepsis



- Syllepsis

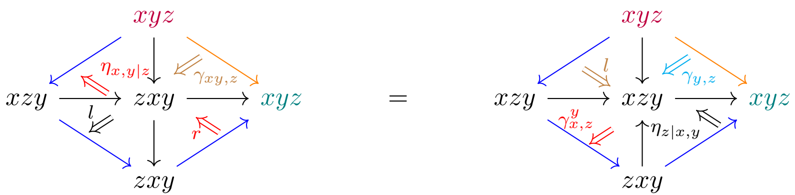


- Symmetry condition:



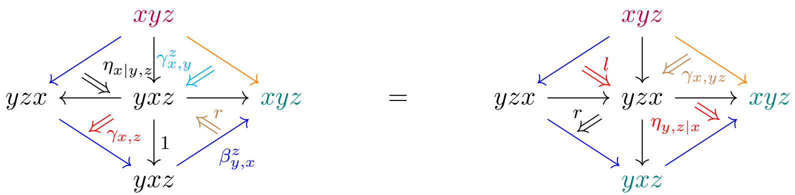
$$\gamma_{x,y}^{\beta_{x,y}} = \gamma_{y,x}.$$

- (2,1)-Syllepsis axiom:



$$\eta_{z|x,y} = \eta_{x,y|z}^{-\beta_{z,xy}} \cdot \gamma_{xy,z}^{-1} \cdot \gamma_{y,z} \cdot (\gamma^{y}_{x,z})^{\beta_{z,y}}.$$

- (1,2)-Syllepsis axiom:



$$\eta_{y,z|x} = \gamma_{x,yz}^{-1} \cdot \gamma^{z}_{x,y} \cdot \gamma_{x,z}^{\beta^{z}_{y.x}} \cdot (\eta_{x|y,z})^{-\beta_{z,x} \cdot \beta^{z}_{y,x}}.$$

- Are these two axioms equivalent?
  - Yes! (Using the symmetry condition).
- Goal
  - Generalize to SM bi-Cat.
- What is so special about these axioms?
  - Cocycles

Group cohomology with values in an Abelian group

Definition 2

Cohomology of a group G with coefficients in an abelian group A is: $H^n(G, A) = H^n(\mathbf{B}G, A) = \pi_0(\mathrm{Hom}_{Top}(\mathbf{B}G, K(A, n)))$.

Group cohomology with values in an Abelian group

Definition 2

Cohomology of a group G with coefficients in an abelian group A is:
$H^n(G, A) = H^n(\mathbf{B}G, A) = \pi_0(\mathrm{Hom}_{Top}(\mathbf{B}G, K(A, n)))$.

- $K(A, n)$ is the Eilenberg-MacLane space.

Group cohomology with values in an Abelian group

Definition 2

Cohomology of a group G with coefficients in an abelian group A is: $H^n(G, A) = H^n(\mathbf{B}G, A) = \pi_0(\mathrm{Hom}_{Top}(\mathbf{B}G, K(A, n)))$.

- $K(A, n)$ is the Eilenberg-MacLane space.
- $\mathbf{B}G$ is the classifying space due to the bar construction ($\bar{W}G$):

$$\cdots \longrightarrow G \times G \times G \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; G \times G \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; G \;\substack{\longrightarrow\\\longrightarrow}\; *$$

  - $\partial_n^0[g_1|\cdots|g_n] = [g_2|\cdots|g_n]^{g_1}$,
  - $\partial_n^i[g_1|\cdots|g_n] = [g_1|\cdots|g_i \cdot g_{i+1}|\cdots|g_n]$ for $1 \leq i \leq n-1$,
  - $\partial_n^n[g_1|\cdots|g_n] = [g_1|\cdots|g_{n-1}]$.
- $\bar{W}G$ is also nerve of the groupoid $G \rightrightarrows *$.

Definition 3

For a given R-module M, a projective resolution is an exact sequence of projective modules P_i's as follows:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Definition 3

For a given R-module M, a projective resolution is an exact sequence of projective modules P_i's as follows:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

- $\mathrm{Hom}(-, D)$ is a left-exact functor.
  - $0 \to A \to B \to C \to 0$ is exact implies $\mathrm{Hom}(A, D) \leftarrow \mathrm{Hom}(B, D) \leftarrow \mathrm{Hom}(C, D) \leftarrow 0$ is exact.

Definition 3

For a given R-module M, a projective resolution is an exact sequence of projective modules P_i's as follows:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

- $\mathrm{Hom}(-, D)$ is a left-exact functor.
  - $0 \to A \to B \to C \to 0$ is exact implies $\mathrm{Hom}(A, D) \leftarrow \mathrm{Hom}(B, D) \leftarrow \mathrm{Hom}(C, D) \leftarrow 0$ is exact.
- For a given projective resolution $P_\bullet \to M$, we get a complex

$$0 \to \mathrm{Hom}(P_0, D) \to \mathrm{Hom}(P_1, D) \to \mathrm{Hom}(P_2, D) \to \cdots$$

- Cohomology groups of this complex are defined as Right-derived functor:

$$R^i\mathrm{Hom}(M, D) := H^i(\mathrm{Hom}(P_\bullet, D)).$$

Definition 3

For a given R-module M, a projective resolution is an exact sequence of projective modules P_i's as follows:

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

- $\mathrm{Hom}(-, D)$ is a left-exact functor.
  - $0 \to A \to B \to C \to 0$ is exact implies $\mathrm{Hom}(A, D) \leftarrow \mathrm{Hom}(B, D) \leftarrow \mathrm{Hom}(C, D) \leftarrow 0$ is exact.
- For a given projective resolution $P_\bullet \to M$, we get a complex

$$0 \to \mathrm{Hom}(P_0, D) \to \mathrm{Hom}(P_1, D) \to \mathrm{Hom}(P_2, D) \to \cdots$$

- Cohomology groups of this complex are defined as Right-derived functor:

$$R^i\mathrm{Hom}(M, D) := H^i(\mathrm{Hom}(P_\bullet, D)).$$

- $H^n(G, A) \cong \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) = H^n\left(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet(\mathbb{Z}), A)\right).$
  - $P_n = F(U(G^{\times n}))$.

Definition 4

A Picard category is a SM Cat such that objects are invertible up to 1-morphisms and 1-morphisms are also invertible.

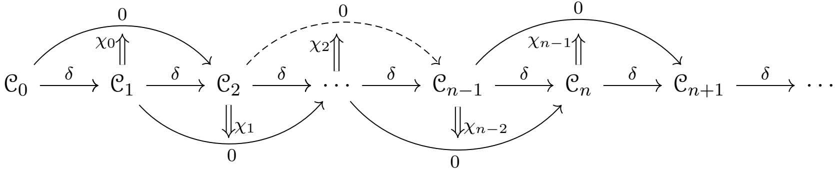
- Picard categories form a SM 2-Cat.

Cohomology of Picard categories

Definition 4

A Picard category is a SM Cat such that objects are invertible up to 1-morphisms and 1-morphisms are also invertible.

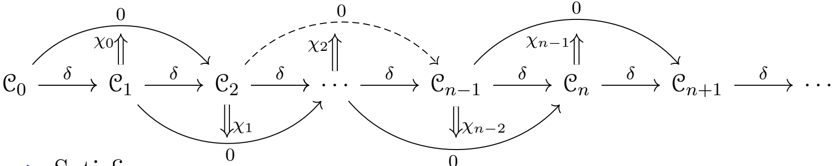
- Picard categories form a SM 2-Cat.
- A complex of Picard categories[2] looks like:



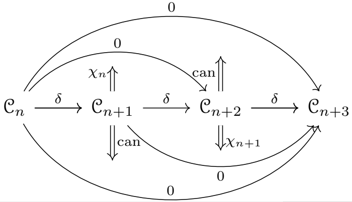
Definition 4

A Picard category is a SM Cat such that objects are invertible up to 1-morphisms and 1-morphisms are also invertible.

- Picard categories form a SM 2-Cat.
- A complex of Picard categories[2] looks like:



  - Satisfies *wave*:



- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.

[1] K.-H. Ulbrich. "Group Cohomology for Picard Categories". In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.
- $\mathcal{P}^n(\mathcal{C}_\bullet) = \{(P, g) | P \in \mathcal{C}_n, g : \delta(P) \to I \in \mathrm{Hom}_{\mathcal{C}_{n+1}}\}$: Category of n-psuedocycles.

[1] K.-H. Ulbrich. "Group Cohomology for Picard Categories". In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.
- $\mathcal{P}^n(\mathcal{C}_\bullet) = \{(P, g) | P \in \mathcal{C}_n, g : \delta(P) \to I \in \mathrm{Hom}_{\mathcal{C}_{n+1}}\}$: Category of n-psuedocycles.
- $\mathcal{L}^n(\mathcal{C}_\bullet) \subseteq \mathcal{P}^n(\mathcal{C}_\bullet)$ for which

$$I_{n+2} \xrightarrow{\chi_n^{-1}} \delta(\delta(P)) \xrightarrow{\delta(g)} \delta(I_{n+1}) \longrightarrow I_{n+2} \qquad = \qquad I_{n+2} \xrightarrow{id} I_{n+2}$$

[1] K.-H. Ulbrich. “Group Cohomology for Picard Categories”. In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.
- $\mathcal{P}^n(\mathcal{C}_\bullet) = \{(P, g) | P \in \mathcal{C}_n, g : \delta(P) \to I \in \mathrm{Hom}_{\mathcal{C}_{n+1}}\}$: Category of n-psuedocycles.
- $\mathcal{L}^n(\mathcal{C}_\bullet) \subseteq \mathcal{P}^n(\mathcal{C}_\bullet)$ for which

$$I_{n+2} \xrightarrow{\chi_n^{-1}} \delta(\delta(P)) \xrightarrow{\delta(g)} \delta(I_{n+1}) \longrightarrow I_{n+2} \qquad = \qquad I_{n+2} \xrightarrow{id} I_{n+2}$$

- $\mathcal{Z}^n(\mathcal{C}_\bullet) =$ Isomorphism classes of objects of $\mathcal{L}^n(\mathcal{C}_\bullet)$.

[1] K.-H. Ulbrich. "Group Cohomology for Picard Categories". In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.
- $\mathcal{P}^n(\mathcal{C}_\bullet) = \{(P, g) | P \in \mathcal{C}_n, g : \delta(P) \to I \in \mathrm{Hom}_{\mathcal{C}_{n+1}}\}$: Category of n-psuedocycles.
- $\mathcal{L}^n(\mathcal{C}_\bullet) \subseteq \mathcal{P}^n(\mathcal{C}_\bullet)$ for which

$$I_{n+2} \xrightarrow{\chi_n^{-1}} \delta(\delta(P)) \xrightarrow{\delta(g)} \delta(I_{n+1}) \longrightarrow I_{n+2} \qquad = \qquad I_{n+2} \xrightarrow{id} I_{n+2}$$

- $\mathcal{Z}^n(\mathcal{C}_\bullet)$ = Isomorphism classes of objects of $\mathcal{L}^n(\mathcal{C}_\bullet)$.
- $\mathcal{B}^n(\mathcal{C}_\bullet) \subseteq \mathcal{Z}^n(\mathcal{C}_\bullet)$ of elements of the form

$$(\delta(Q), \chi_Q), Q \in \mathcal{C}_{n-1}.$$

- $H^n(\mathcal{C}_\bullet) = \mathcal{Z}^n(\mathcal{C}_\bullet)/\mathcal{B}^n(\mathcal{C}_\bullet)$.[1]

[1] K.-H. Ulbrich. "Group Cohomology for Picard Categories". In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

Group cohomology with values in a Picard category

- Any set X, category $\mathcal{C}$: $\mathcal{C}^X = \mathrm{Funct}\,(X \to \mathcal{C})$
  - X as a discrete category, i.e., $\mathrm{Ob} = X$, $\mathrm{Mor} = \mathrm{id}$.

Group cohomology with values in a Picard category

- Any set X, category $\mathcal{C}$: $\mathcal{C}^X = \text{Funct}\,(X \to \mathcal{C})$
  - X as a discrete category, i.e., $\text{Ob} = X$, $\text{Mor} = \text{id}$.
- Given any sSet $X_\bullet$, consider $\mathcal{C}^{X_\bullet}$

$$\mathcal{C}^{X_0} \rightrightarrows \mathcal{C}^{X_1} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{C}^{X_2} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \cdots$$

  - $d_n^{i^*} : \mathcal{C}^{X_n} \to \mathcal{C}^{X_{n+1}} : (X_n \xrightarrow{f} \mathcal{C}) \mapsto (X_{n+1} \xrightarrow{d_{n+1}^i} X_n \xrightarrow{f} \mathcal{C})$

Group cohomology with values in a Picard category

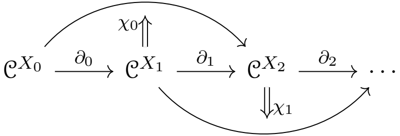
- Any set X, category $\mathcal{C}$: $\mathcal{C}^X = \text{Funct}\,(X \to \mathcal{C})$
  - X as a discrete category, i.e., Ob $= X$, Mor $=$ id.
- Given any sSet $X_\bullet$, consider $\mathcal{C}^{X_\bullet}$

  $$\mathcal{C}^{X_0} \rightrightarrows \mathcal{C}^{X_1} \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{C}^{X_2} \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \cdots$$

  - $d_n^{i^*} : \mathcal{C}^{X_n} \to \mathcal{C}^{X_{n+1}} : (X_n \xrightarrow{f} \mathcal{C}) \mapsto (X_{n+1} \xrightarrow{d_{n+1}^i} X_n \xrightarrow{f} \mathcal{C})$
- By taking the alternating sum

$$\partial_n = \sum_{i=0}^{n+1} (-1)^i d_n^{i^*}$$

- We get a cochain complex:



- We can calculate cohomology $H^n(X_\bullet, \mathcal{C})$ for this.

Application to SM Cat

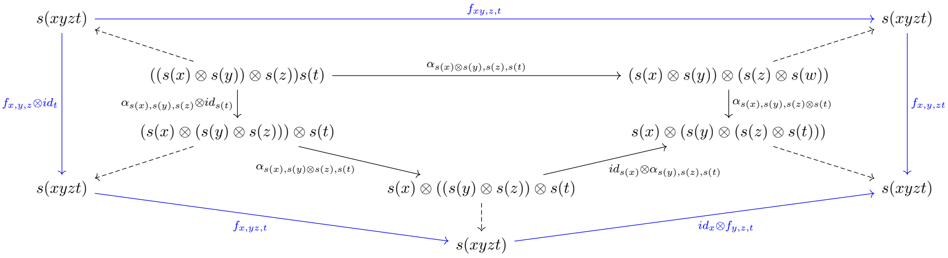
- SM Cat $\mathcal{E}$.
- $G = \pi_0(\mathcal{E}) = \mathrm{Ob}(\mathcal{E})/\cong$. (Group).
- $A = \pi_1(\mathcal{E}) = \mathrm{Aut}_{\mathcal{E}}(I)$. (Abelian Group).

Application to SM Cat

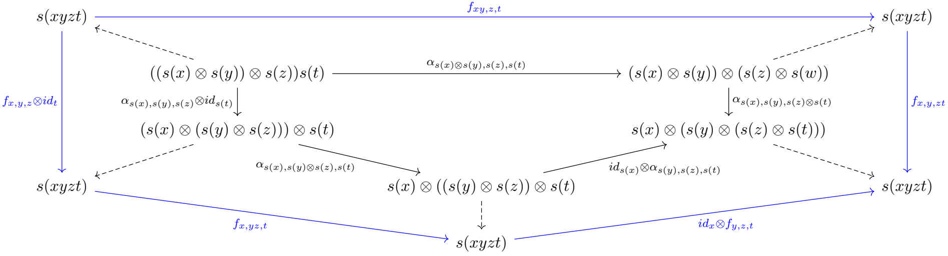
- SM Cat $\mathcal{E}$.
- $G = \pi_0(\mathcal{E}) = \mathrm{Ob}(\mathcal{E})/\cong$. (Group).
- $A = \pi_1(\mathcal{E}) = \mathrm{Aut}_{\mathcal{E}}(I)$. (Abelian Group).
- $s : G \to \mathcal{E}$ be an arbitrary section of $p : \mathcal{E} \to G$.
  - For each $x, y \in G, \exists\, c_{x,y} : s(x) \otimes s(y) \xrightarrow{\cong} s(xy)$ along with

$$\begin{array}{ccccc}
(s(x) \otimes s(y)) \otimes s(z) & \xrightarrow{c_{x,y} \otimes id_{s(z)}} & s(xy) \otimes s(z) & \xrightarrow{c_{xy,z}} & s(xyz) \\
\alpha_{s(x),s(y),s(z)} \big\downarrow & & & & \big\downarrow f_{x,y,z} \\
s(x) \otimes (s(y) \otimes s(z)) & \xrightarrow[id_{s(x)} \otimes c_{y,z}]{} & s(x) \otimes s(yz) & \xrightarrow[c_{x,yz}]{} & s(xyz)
\end{array}$$

- Such that it satisfies the pentagon:



- Such that it satisfies the pentagon:



- $f \in H^3(G, A)$.
  - f satisfies the cocycle condition due to the pentagon.

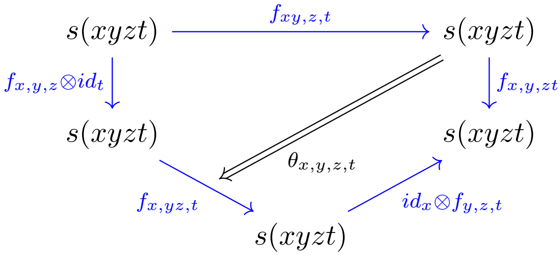
Application to SM bi-Cat

- SM bi-Cat $\mathbb{E}$.
- $G = \pi_0(\mathbb{E}) = \mathrm{Ob}(\mathbb{E})/\cong$. (Group).
- $\mathcal{A} = \Pi_1(\mathbb{E}) = \mathrm{Aut}_{\mathbb{E}}(I)$. (Picard category).

- SM bi-Cat $\mathbb{E}$.
- $G = \pi_0(\mathbb{E}) = \mathrm{Ob}(\mathbb{E})/\cong$. (Group).
- $\mathcal{A} = \Pi_1(\mathbb{E}) = \mathrm{Aut}_{\mathbb{E}}(I)$. (Picard category).
- $s : G \to \mathbb{E}$ be an arbitrary section of $p : \mathbb{E} \to G$.
  - For each $x, y \in G$, $\exists\ c_{x,y} : s(x) \otimes s(y) \xrightarrow{\cong} s(xy)$ along with

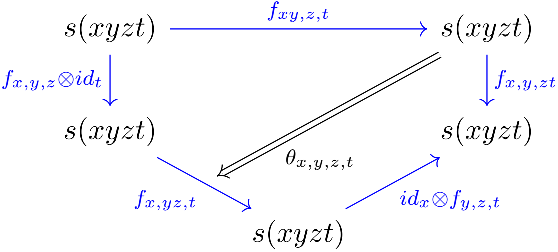
$$
\begin{array}{ccccc}
(s(x) \otimes s(y)) \otimes s(z) & \xrightarrow{c_{x,y} \otimes id_{s(z)}} & s(xy) \otimes s(z) & \xrightarrow{c_{xy,z}} & s(xyz) \\
\alpha_{s(x),s(y),s(z)} \downarrow & & \varphi_{x,y,z} \ (\Leftarrow) & & \downarrow f_{x,y,z} \\
s(x) \otimes (s(y) \otimes s(z)) & \xrightarrow[id_{s(x)} \otimes c_{y,z}]{} & s(x) \otimes s(yz) & \xrightarrow[c_{x,yz}]{} & s(xyz)
\end{array}
$$

- And the pentagon:

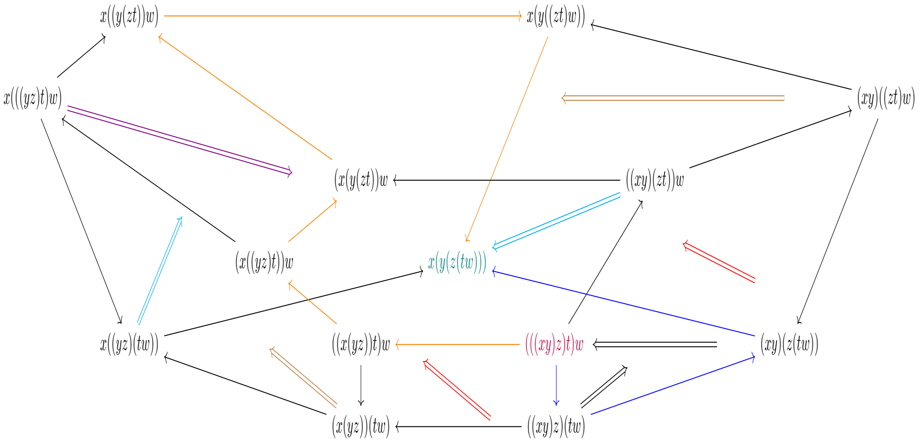


- Such that this satisfies the *associahedron* K_5.

- And the pentagon:



- Such that this satisfies the *associahedron* K_5.
  - Compatibility with 5 variables.



Associahedron K_5 [2]

[2] Jean-Louis Loday. "The diagonal of the Stasheff polytope". In: *Higher structures in geometry and physics, Volume 287* (2011), pp. 269–292.

- $\mathcal{P}^n(\mathcal{C}_\bullet) = \{(P, g) | P \in \mathcal{C}_n, g : \delta(P) \to I \in \mathrm{Hom}_{\mathcal{C}_{n+1}}\}$: Category of n-psuedocycles.
- $\mathcal{L}^n(\mathcal{C}_\bullet) \subseteq \mathcal{P}^n(\mathcal{C}_\bullet)$ for which

$$I_{n+2} \xrightarrow{\chi_n^{-1}} \delta(\delta(P)) \xrightarrow{\delta(g)} \delta(I_{n+1}) \longrightarrow I_{n+2} \qquad = \qquad I_{n+2} \xrightarrow{id} I_{n+2}$$

- $\mathcal{Z}^n(\mathcal{C}_\bullet)$ = Isomorphism classes of objects of $\mathcal{L}^n(\mathcal{C}_\bullet)$.

- $(f,\theta) \in H^3(G,\mathcal{A})$.
  - $f \in \mathcal{C}_3 = \mathcal{A}^{G_3} = \mathcal{A}^{G\times G\times G}$. (Bar construction).
  - $\theta \in \mathrm{Hom}_{\mathcal{C}_4} = \mathrm{Hom}_{\mathcal{A}^{G_4}} = \mathrm{Hom}_{\mathcal{A}^{G\times G\times G\times G}}$.
  - $(f,\theta) \in \mathcal{L}^3(\mathcal{C}_\bullet) = \mathcal{L}^3\left(\mathcal{A}^{G_\bullet}\right)$ due to the associahedron K_5.

Long exact seuence of cohomology groups

$$\cdots \longrightarrow H^{n+1}(G, \pi_1(\mathcal{A})) \longrightarrow H^n(G, \mathcal{A}) \longrightarrow H^n(G, \pi_0(\mathcal{A})) \longrightarrow H^{n+2}(G, \pi_1(\mathcal{A})) \longrightarrow \cdots$$

For $n = 3$: $[f, \theta] \longmapsto [f]$

What about braiding?

- Already too complicated just with the associator (α).
- For braiding, we need a different tool.

What about braiding?

- Already too complicated just with the associator (α).
- For braiding, we need a different tool.

Definition 5

- E is an extension of B by A if there is a short exact sequence

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0.$$

- If $A \to E$ factors through the center of E, it is called a central extension.

What about braiding?

- Already too complicated just with the associator (α).
- For braiding, we need a different tool.

Definition 5

- E is an extension of B by A if there is a short exact sequence

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0.$$

- If $A \to E$ factors through the center of E, it is called a central extension.

- $$H^2(B,A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} CentrExt(B,A) \quad (\cong_{\mathbf{Grp}})$$

What about braiding?

- Already too complicated just with the associator (α).
- For braiding, we need a different tool.

Definition 5

- E is an extension of B by A if there is a short exact sequence

$$0 \to A \xrightarrow{i} E \xrightarrow{p} B \to 0.$$

- If $A \to E$ factors through the center of E, it is called a central extension.

- $$H^2(B, A) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} CentrExt(B, A) \quad \cong_{\mathbf{Grp}}$$
- $[c] \in H^2(B, A)$: $c : B \times B \to A$. Define a group with $U(G) \times U(A)$,

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 \cdot b_2, a_1 + a_2 + c(b_1, b_2)).$$

- Choose a section $s : U(B) \to U(E)$, define

$$c(b_1, b_2) = s(b_1)s(b_2)s(b_1 b_2)^{-1}.$$

Extensions		Cohomology
$0 \to A \to E \to G \to 0$	$\longleftrightarrow$	$H^2(G, A)$
$0 \to \mathcal{A} \to \mathcal{E} \to G \to 0$	$\longleftrightarrow$	$H^3(G, A)$

- $\mathcal{E}$ is a SM Cat.
- $G = \pi_0(\mathcal{E})$.
- $\mathcal{A}$ is the groupoid corresponding to $A = \pi_1(\mathcal{E}) = \mathrm{Aut}_{\mathcal{E}}(I)$.
    - $\mathcal{A} = \Sigma A$.

For braiding...

Theorem 6 (Universal Coefficient Theorem)

For an abelian group G and a trivial G-module A, there exists a split short exact sequence:

$$0 \to Ext^1(H_{n-1}(G), A) \to H^n(G, A) \to Hom_{\boldsymbol{Ab}}(H_n(G), A) \to 0.$$

For braiding...

Theorem 6 (Universal Coefficient Theorem)

For an abelian group G and a trivial G-module A, there exists a split short exact sequence:

$$0 \to Ext^1(H_{n-1}(G), A) \to H^n(G, A) \to Hom_{\boldsymbol{Ab}}(H_n(G), A) \to 0.$$

- $L_p \Lambda^q B \to H_{p+q}(B)$.
  - L is a left-derived functor, Λ is the exterior power.

For braiding...

Theorem 6 (Universal Coefficient Theorem)

For an abelian group G and a trivial G-module A, there exists a split short exact sequence:

$$0 \to Ext^1(H_{n-1}(G), A) \to H^n(G, A) \to Hom_{\boldsymbol{Ab}}(H_n(G), A) \to 0.$$

- $L_p \Lambda^q B \to H_{p+q}(B)$.
  - L is a left-derived functor, Λ is the exterior power.
- $p = 0$: $\Lambda^q B \to H_q(B)$ give us extensions.
- $p = 1$: $L\Lambda^q B \to H_{q+1}(B)$ give us biextensions.

For braiding...

Theorem 6 (Universal Coefficient Theorem)

For an abelian group G and a trivial G-module A, there exists a split short exact sequence:

$$0 \to Ext^1(H_{n-1}(G), A) \to H^n(G, A) \to Hom_{\boldsymbol{Ab}}(H_n(G), A) \to 0.$$

- $L_p \Lambda^q B \to H_{p+q}(B)$.
  - L is a left-derived functor, Λ is the exterior power.
- $p = 0$: $\Lambda^q B \to H_q(B)$ give us extensions.
- $p = 1$: $L\Lambda^q B \to H_{q+1}(B)$ give us biextensions.

Extensions		Cohomology		Biextensions
$A \to E \to G$	$\longleftrightarrow$	$H^2(G, A)$		
$\mathcal{A} \to \mathcal{E} \to G$	$\longleftrightarrow$	$H^3(G, A)$	$\longleftrightarrow$	$A \to E \to G \times G$

- $E_{x,y} = \mathrm{Hom}_{\mathcal{E}}(XY, YX)$.

- For a SM bi-Cat $\mathbb{E}$:

Extensions		Cohomology		Biextensions[3]
$A \to E \to G$	$\longleftrightarrow$	$H^2(G, A)$		
$\mathcal{A} \to \mathcal{E} \to G$	$\longleftrightarrow$	$H^3(G, A)$	$\longleftrightarrow$	$A \to E \to G \times G$
$\mathbb{A} \to \mathbb{E} \to G$	$\longleftrightarrow$	$H^3(G, \mathcal{A})$	$\longleftrightarrow$	$\mathcal{A} \to \mathcal{E} \to G \times G$

- $\mathbb{A} = \Sigma\mathcal{A} = \Sigma^2 A$.
- $\mathcal{E}_{x,y} = \mathcal{H}\mathrm{om}_{\mathbb{E}}(XY, YX)$ along with the contracted product:
  - $\otimes_1 : \mathcal{E}_{x,y} \wedge^A \mathcal{E}_{x',y} \to \mathcal{E}_{xx',y}$.
  - $\otimes_2 : \mathcal{E}_{x,y} \wedge^A \mathcal{E}_{x,y'} \to \mathcal{E}_{x,yy'}$.

[3] Lawrence Breen. "Monoidal Categories and Multiextensions". In: *Compositio Mathematica Volume 117* (1999), pp. 295–335.

- For a SM bi-Cat $\mathbb{E}$:

Extensions		Cohomology		Biextensions[3]
$A \to E \to G$	$\longleftrightarrow$	$H^2(G, A)$		
$\mathcal{A} \to \mathcal{E} \to G$	$\longleftrightarrow$	$H^3(G, A)$	$\longleftrightarrow$	$A \to E \to G \times G$
$\mathbb{A} \to \mathbb{E} \to G$	$\longleftrightarrow$	$H^3(G, \mathcal{A})$	$\longleftrightarrow$	$\mathcal{A} \to \mathcal{E} \to G \times G$

- $\mathbb{A} = \Sigma\mathcal{A} = \Sigma^2 A$.
- $\mathcal{E}_{x,y} = \mathcal{H}om_{\mathbb{E}}(XY, YX)$ along with the contracted product:
  - $\otimes_1 : \mathcal{E}_{x,y} \wedge^A \mathcal{E}_{x',y} \to \mathcal{E}_{xx',y}$.
  - $\otimes_2 : \mathcal{E}_{x,y} \wedge^A \mathcal{E}_{x,y'} \to \mathcal{E}_{x,yy'}$.
- Work in progress...

[3] Lawrence Breen. "Monoidal Categories and Multiextensions". In: *Compositio Mathematica Volume 117* (1999), pp. 295–335.

Thank you!

References I

[1] Ronald Brown and İlhan İçen. "Homotopies and Automorphisms of Crossed Modules of Groupoids". In: *Applied Categorical Structures* (2003), p. 193.

[2] Michael Horst. *Cohomology of Picard Categories*. The Ohio State University, 2020, pp. 54–62.

[3] K.-H. Ulbrich. "Group Cohomology for Picard Categories". In: *Journal of Algebra, Volume 91* (1984), pp. 464–498.

[4] Jean-Louis Loday. "The diagonal of the Stasheff polytope". In: *Higher structures in geometry and physics, Volume 287* (2011), pp. 269–292.

[5] Lawrence Breen. "Monoidal Categories and Multiextensions". In: *Compositio Mathematica Volume 117* (1999), pp. 295–335.

Cochain complex of Picard categories

- Definition of Relative Kernel and Cokernel.
- $H^n(\mathcal{C}_\bullet) \in \mathbf{Pic}$.