Stabilization of 2-Crossed Modules

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Introduction

Gadget

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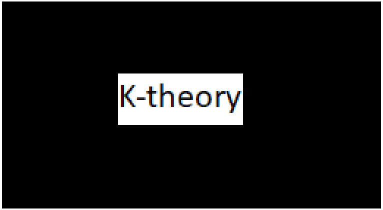

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Space with interesting homotopy groups

- Examples of such gadgets.
  - Category of finitely generated projective R-modules. (As discussed by Niles Johnson). If X is the output of K-theory on finitely generated projective R-modules, then we have
    - ★ $\pi_1(X) = K_0(R)$.
    - ★ $\pi_2(X) = K_1(R) = R^{\times}$ = Units of R.
  - A Waldhausen Category.

Waldhausen category

A Waldhausen category[a] $\mathcal{C}$ is a category with a zero object, 0 equipped with two classes of morphisms: weak equivalences (WE) and cofibrations (CO) such that it has a notion of taking quotients, and satisfy certain conditions.

[a] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory.* American Mathematical Society, 2010, pp. 172–174.

Examples of Waldhausen categories

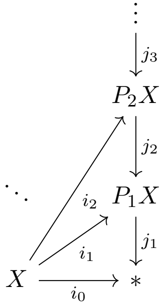
1. The category of finite sets, with inclusions as cofibrations, and isomorphisms as weak equivalences.
2. The category **R-Mod**, for any ring R with the injective maps as the cofibrations, and isomorphisms as the weak equivalences.
3. In fact, any exact category, hence any abelian category is naturally a Waldhausen category with monomorphisms as the cofibrations and isomorphisms as the weak equivalences.

Examples of Waldhausen categories

- The category of bounded below ($k \geq 0$) chain complexes over a ring R, $\mathbf{Ch}_R$ with $f : M \to N \in Hom_{Ch_R}(M, N)$ to be
  - a weak equivalence if f induces isomorphism on homology groups.
  - a cofibration if for each $k \geq 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective module as its cokernel.

n-Types

The resulting space of K-theory is very complicated to analyse. So, we break it down in n-types.



n-Types

n-type[a] is the full subcategory of $\text{Top}^*/\simeq$ (i.e., pointed topological spaces up to homotopy equivalence) consisting of connected CW-spaces Y with $\pi_i(Y) = 0$ for $i > n$.

[a] Hans-Joachim Baues. "Combinatorial Homotopy and 4-Dimensional Complexes". In: *Walter de Gruyter* (1991), pp. 171–177.

Motivation

- From a given Waldhausen category, it is well know that we can get a 1-type and a 2-type.
- Now, for a given Waldhausen category, we want to find a 3-type.

Models for n-types

To analyse these n-types we study corresponding algebraic models, and we further relate them with n-Categories as follows:

Example of a 1-type

Groups can be considered as algebraic models for the 1-type.
For a given space X such that,

$$\pi_i(X) = \begin{cases} G & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}$$

define the space $BG := |N(G \rightrightarrows *)|$
Then we get $X \simeq BG$.

Nerve of a category

Nerve of a small category $\mathcal{C}$ is a simplicial complex $N_\bullet(\mathcal{C})$.

- $N_0(\mathcal{C}) = 0\text{-cells} = Ob(\mathcal{C})$:

$$\bullet A$$

- $N_1(\mathcal{C}) = 1\text{-cells} = \text{Morphisms of } \mathcal{C}$:

$$A_1 \xrightarrow{f} A_2$$

- $N_2(\mathcal{C}) = 2\text{-cells} = \text{A pair of composable morphisms in } \mathcal{C}$:

$$\begin{array}{ccccc} & & A_3 & & \\ & \overset{f_2 \circ f_1}{\nearrow} & & \overset{f_2}{\nwarrow} & \\ A_1 & & \xrightarrow[f_1]{} & & A_2 \end{array}$$

  i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$.

- $N_k(\mathcal{C}) = k\text{-cells} = k\text{-composable morphisms}$, i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{k-1}} A_k \xrightarrow{f_k} A_{k+1}$.

Example of a 1-type

Groups can be considered as algebraic models for the 1-type.

- For a given space X such that,

$$\pi_i(X) = \begin{cases} G & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases}$$

  define the space $BG := |N(G \rightrightarrows *)|$
  Then we get $X \simeq BG$.

- So given a space X with π_1 as the only one non-trivial homotopy group, we can construct a category $\mathcal{G}$ which can represent X up to homotopy equivalence.

$$X \simeq |N\mathcal{G}|$$

- We consider the group G as a corresponding algebraic model.
- And we consider the category $\mathcal{G}$ as a corresponding categorical model.

Theorem 1 (Homotopy Hypothesis (Grothendieck))

By taking classifying spaces and fundamental n-groupoids, there is an equivalence between the theory of weak n-goupoids and that of homotopy n-types.

n-types	Categorical model	Algebraic model	Groups
0-type	0-category	Set	
1-type	1-category	Group	1 group
2-type	2-category	Crossed module	2 groups
3-type	3-category	2-Crossed module	3 groups

Crossed Module

Crossed module

A crossed module[a] G_* consists of a G_0-equivariant group homomorphism, where G_0 acts on itself by conjugation.

$$G_1 \xrightarrow{\partial} G_0$$

where the action of G_0 on G_1 satisfies

- $f^{\partial g} = g^{-1} f g$.

[a] H.-J. Baues and Daniel Conduché. "On the 2-type of an iterated loop space". In: *Forum Mathematicum* (1997), pp. 725–733.

Remark

The homotopy groups of the crossed module G_* are:

- $\pi_0(G_*) = \mathrm{Coker}\partial$,
- $\pi_1(G_*) = \mathrm{Ker}\partial$.

Extending the previous idea for higher values of n:

$$X \simeq |N\mathcal{G}| \tag{1}$$

- $n = 2$. For a given Crossed module G_*, we can construct a category $\Gamma(G_*)$ such that
  - Ob $(\Gamma(G_*)) = G_0$
  - 1-Mor $(\Gamma(G_*)) = G_0 \rtimes G_1$
    - ⋆ G_1 acts on G_0 by sending $x_0 \mapsto x_0 \cdot \partial f$ for $f \in G_1$.
- For equation 1, $\mathcal{G} = (\Gamma(G_*) \rightrightarrows *)$ works.

2-Crossed Module

2-Crossed Module

A 2-crossed module[a] G_* consists of a complex of G_0-groups

$$\begin{array}{c} G_1 \times G_1 \\ \downarrow \{\cdot,\cdot\} \\ G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0 \end{array}$$

(so that $\partial\partial = 0$) and ∂'s are G_0-equivariant, where G_0 acts on itself by conjugation, such that $G_2 \xrightarrow{\partial} G_1$ is a crossed module such that

- $(\alpha^f)^x = (\alpha^x)^{f^x}$ for all $\alpha \in G_2, f \in G_1, x \in G_0$.
- There is a function $\{\cdot,\cdot\} : G_1 \times G_1 \to G_2$ called Peiffer lifting.
- Compatibility conditions.

[a] Ronald Brown and İlhan İçen. "Homotopies and Automorphisms of Crossed Modules of Groupoids". In: *Applied Categorical Structures* (2003), p. 193.

Remark

The homotopy groups of a 2-crossed module G_* are:

- $\pi_0(G_*) = \text{Coker}(\partial : G_1 \to G_0)$,
- $\pi_1(G_*) = \text{Ker}(\partial : G_1 \to G_0)/(\text{Im}(\partial : G_2 \to G_1))$,
- $\pi_2(G_*) = \text{Ker}(\partial : G_2 \to G_1)$.

Remark

- The groups defined above are well-defined.
- $\pi_1(G_*)$, $\pi_2(G_*)$ are abelian. (So, they could be seen as corresponding homotopy groups of a space).

Extending the previous idea for higher values of n:

$$X \simeq |N\mathcal{G}|$$

- $n = 2$. For a given Crossed module G_*, we can construct a category $\Gamma(G_*)$ such that
  - Ob $(\Gamma(G_*)) = G_0$
  - 1-Mor $(\Gamma(G_*)) = G_0 \rtimes G_1$
    - G_1 acts on G_0 by sending $x_0 \mapsto x_0 \cdot \partial f$ for $f \in G_1$.
- For equation 1, $\mathcal{G} = (\Gamma(G_*) \rightrightarrows *)$ works.
- $n = 3$. Now, for 2-Crossed modules, we extend this logic and construct a 2-Category structure, and later we try to stabilize it by putting a kind of commutative group law (i.e., a symmetric monoidal structure).

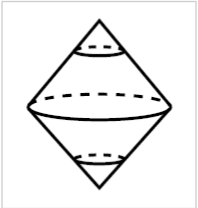
Stability

Stability

- We want to make the 2-Crossed modules stable.
- The output of K-theory is in fact a spectrum $\mathbb{X} = \{X_n\}_{n\geq 0}$, $\Sigma X_n \to X_{n+1}$.

Suspension

For a space X, the suspension ΣX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. $(\Sigma X = S^1 \wedge X)$.



Example: $\Sigma S^n = S^{n+1}$

Theorem 2 (Freudenthal Suspension Theorem)

For a spectrum $\mathbb{X} = \{X_n\}_{n\geq 0}$, *the sequence*

$$\pi_i(X_n) \to \pi_{i+1}(X_{n+1}) \to \pi_{i+2}(X_{n+2}) \to \cdots$$

eventually stabilizes.

Stable Homotopy Group

The i^{th} stable homotopy group of $\mathbb{X}$ is:

$$\pi_i^s(\mathbb{X}) = \lim_{\overrightarrow{k}} \pi_{i+k}(X_k) \cong \pi_{i+N}(X_N),\ N \gg 0$$

Theorem 3 (The Stable Homotopy Hypothesis)

Symmetric monoidal structure corresponds to topological stability.

$$
\begin{array}{ccccc}
\text{Stable 1-types} & \longleftrightarrow & \text{Symmetric Monoidal Categories} & \longleftrightarrow & \text{Stable Crossed Module} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Stable 2-types} & \longleftrightarrow & \text{Symmetric Monoidal 2-Categories} & \longleftrightarrow & \text{Stable 2-Crossed Modules} \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}
$$

SM 2-Cat structure on a 2-CM

- Given a 2-CM G_*

$$G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0$$

We can construct a 2-Cat $\Gamma(G_*)$:

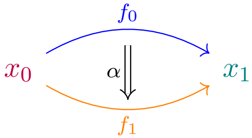
- $Ob(\Gamma(G_*)) = G_0$.

$$x_0 \in G_0.$$

- 1-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1$.

$$x_0 \xrightarrow{f_0} x_1 \text{ such that } x_1 = x_0 \cdot \partial(f_0).$$

- 2-Mor$(\Gamma(G_*)) = G_0 \rtimes G_1 \rtimes G_2$.



Such that $f_1 = f_0 \cdot \partial(\alpha)$.

Compositions of 2-cells:

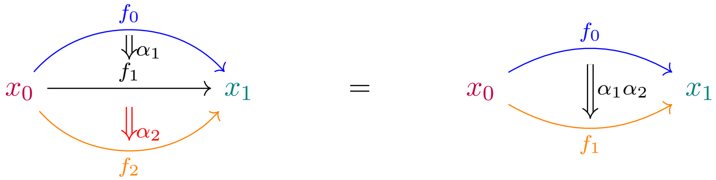


Figure 1: Vertical composition

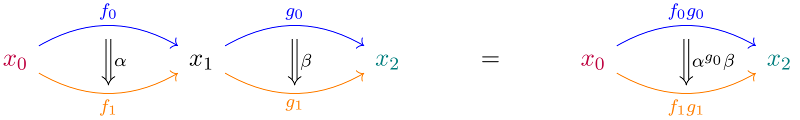


Figure 2: Horizontal composition

They satisfy certain compatibility conditions.

Theorem 4 (Eckmann–Hilton argument)

If a set is equipped with two monoid structures with unit objects, such that each one is a homomorphism for the other, then the two structures coincide and the resulting monoid is commutative.

- A group G is abelian if and only if $m : G \times G \to G$ is a homomorphism. Taking motivation from here, we can say:
- Defining a monoidal functor $\_ \otimes \_ : \Gamma(G_*) \times \Gamma(G_*) \to \Gamma(G_*)$ of 2-Categories, would give us stabilization, i.e., a symmetric monoidal structure.

Components of a Symmetric Monoidal 2-Category[1] (SM 2-Cat) are:

- A 2-Cat
- Monoidal structure ($\otimes$) on the 2-Cat
- Braiding (β) on the monoidal structure
- Left ($\eta_{-|--}$) and right ($\eta_{--|-}$) hexagonators
- Syllepsis (γ) (Exclusive for 2-Cat)

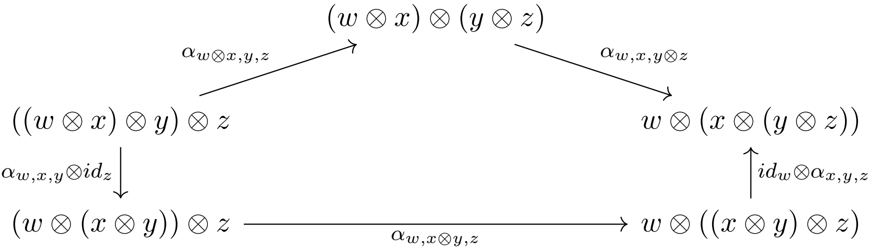
[1] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, 2021, pp. 384–396.

Definition 5

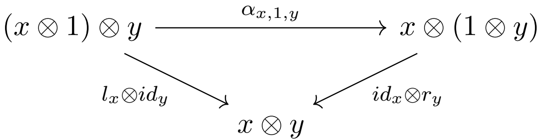
A monoidal category is a category $\mathcal{M}$ equipped with the following data:

1. an object $1 \in \mathrm{Ob}(\mathcal{M})$, called the unit object.
2. a functor: $\_ \otimes \_ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ called the tensor product.
3. a natural isomorphism $\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z)$, called the associator.
4. a natural isomorphism $l_x : x \otimes 1 \xrightarrow{\cong} x$, called the left unitor.
5. a natural isomorphism $r_x : 1 \otimes x \xrightarrow{\cong} x$, called the right unitor such that the following diagrams commute:

- Pentagon identity



- Triangle identity



- Monoidal structure of 2-Crossed modules:

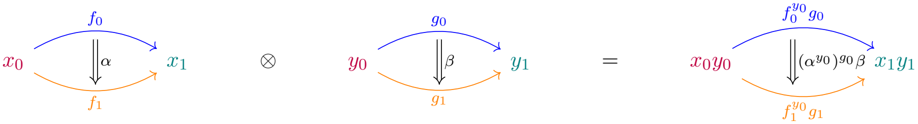
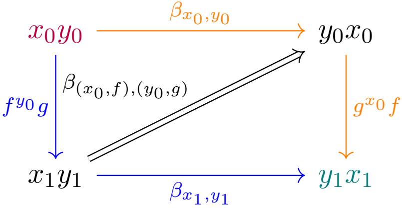


Figure 3: Monoidal structure

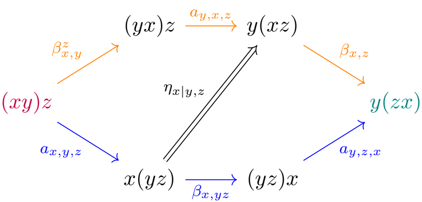
- This is a strict monoidal structure, which means associators $(\alpha_{x,y,z})$, and unitors (l_x, r_x) are identities, and hence the pentagon and triangle identities are also trivially satisfied.

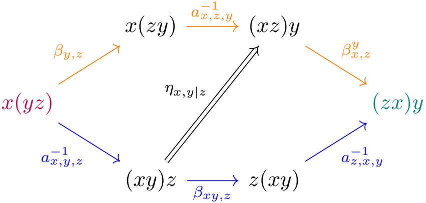
- Braiding:

For every $x_0 \xrightarrow{f} x_1$ and $y_0 \xrightarrow{g} y_1$, we have

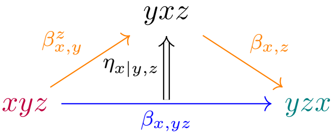


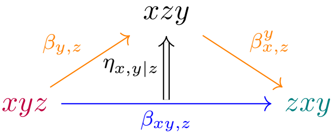
- Left ($\eta_{x|y,z}$) and right ($\eta_{x,y|z}$) hexagonators:





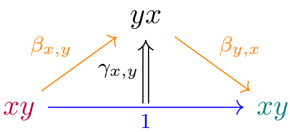
But since we have strict 2-category, we get:



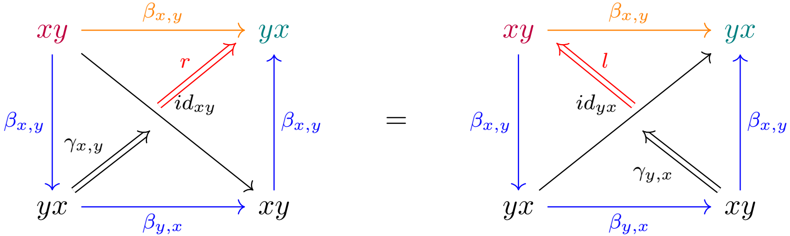


All these satisfy naturality and certain compatibility conditions.

- Syllepsis
  Given any two $x, y \in G_0$, we have



- For 1-Cat this 2-cell collapses to 1.
- They satisfy naturality and certain compatibility conditions and the following condition.
- Symmetry condition:



Current work

- From a given Waldhausen category, it is well know that we can get a group, which is a 1-type, and a Stable Crossed module, which is a 2-type.
- Now, for a given Waldhausen category $\mathcal{C}$, we want to find a 3-type. So we are using the same procedure to get the 2-Crossed module G_*.

- The generators for G_0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for G_1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any WE.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- The generators for G_2 are:
  - $$\begin{array}{ccc} & A_2 & \\ {\scriptstyle\sim}\nearrow & & \nwarrow{\scriptstyle\sim} \\ A_0 & \xrightarrow{\sim} & A_1 \end{array}$$
  - $$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & C_0 \\ {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & C_1 \end{array}$$
  - $$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow\!\uparrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow\!\uparrow & & \uparrow\!\uparrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$
- But this is not stable yet. So we make it stable by realizing the SM 2-Cat structure on it.

References I

[1] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory*. American Mathematical Society, 2010, pp. 172–174.

[2] Hans-Joachim Baues. “Combinatorial Homotopy and 4-Dimensional Complexes”. In: *Walter de Gruyter* (1991), pp. 171–177.

[3] H.-J. Baues and Daniel Conduché. “On the 2-type of an iterated loop space”. In: *Forum Mathematicum* (1997), pp. 725–733.

[4] Ronald Brown and İlhan İçen. “Homotopies and Automorphisms of Crossed Modules of Groupoids”. In: *Applied Categorical Structures* (2003), p. 193.

[5] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, 2021, pp. 384–396.

References II

[6] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Thank You!

Waldhausen category

A Waldhausen category[a] $\mathcal{C}$ is a category with a zero object, 0 equipped with two classes of morphisms: weak equivalences (WE) and cofibrations (CO) such that it has a notion of taking quotients, and satisfy certain conditions.

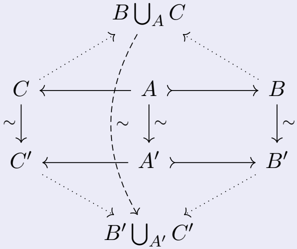
- $\text{iso}(\mathcal{C}) \subseteq \text{WE}(\mathcal{C}) \cap \text{CO}(\mathcal{C})$.
- $0 \to X \in \text{CO}(\mathcal{C})$ for all $X \in \text{Ob}(\mathcal{C})$.
- If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \bigcup_A C$ of these two maps exists in $\mathcal{C}$ and $C \rightarrowtail B \bigcup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \bigcup_A C \end{array}$$

[a] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory.* American Mathematical Society, 2010, pp. 172–174.

Waldhausen category

- Gluing axiom:



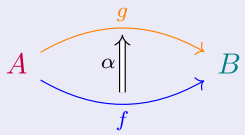
The induced map $B \bigcup_A C \to B' \bigcup_{A'} C'$ is also a weak equivalence.

2-Categories

Definition 6

A (strict) 2-category $\mathcal{C}$ is comprised of the following:

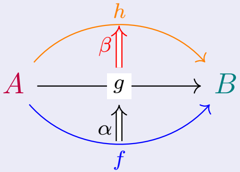
- 0-Cells (Objects): Denoted by $Ob(\mathcal{C})$.
- 1-Cells (Morphisms): For $A, B \in Ob(\mathcal{C})$, a set $\text{Hom}(A, B)$ of 1-cells from A to B, also known as morphisms. A 1-cell is often written textually as $f : A \to B$ or graphically as $A \xrightarrow{f} B$.
- 2-Cells: For $A, B \in Ob(\mathcal{C})$, $f, g \in \text{Hom}(A, B)$, a set $\text{Face}(f, g)$ of 2-cells from f to g. A 2-cell is often written textually as $\alpha : f \Rightarrow g : A \to B$ or graphically as follows:



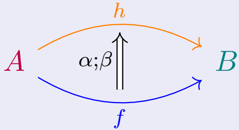
- 1-Composition: For each chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, a 1-cell $A \xrightarrow{f;g} C$.

Definition 6

- Vertical 2-Composition: For a chain of 2-cells



, a 2-cell



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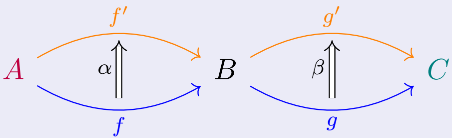
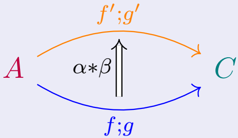
- Horizontal 2-Composition: For each chain of 2-cells



, a 2-cell



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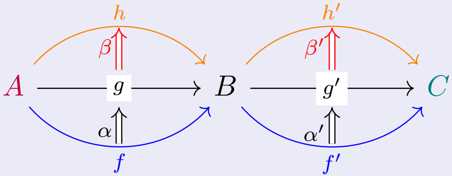
Definition 6

- Associativity: For all the compositions.
- Identities of 1-cells and 2-cells exist and are compatible with all the compositions.
- 2-Interchange: Every clover of 2-cells



satisfies

$(\alpha;\beta) * (\alpha';\beta') = (\alpha * \alpha');(\beta * \beta')$.

Stable Crossed Module

Definition 7

A stable crossed module (SCM)[a] G_* is a crossed module $\partial : G_1 \to G_0$ together with a map Back to main

$$\langle \cdot, \cdot \rangle : G_0 \times G_0 \to G_1$$

satisfying the following for any $f, g \in G_1, x, y, z \in G_0$:

1. $\partial\langle x, y\rangle = [y, x]$,
2. $f^x = f + \langle x, \partial(f)\rangle$,
3. $\langle x, y + z\rangle = \langle x, y\rangle^z + \langle x, z\rangle$,
4. $\langle x, y\rangle + \langle y, x\rangle = 0$.

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[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Simplicial Set

A simplicial set $X \in \mathbf{sSet}$ is

- for each $n \in \mathbb{N}$ a set $X_n \in \mathbf{Set}$ (the set of n-simplices),
- for each injective map $\partial_i : [n-1] \to [n]$ of totally ordered sets ($[n] \colon = (0 < 1 < \cdots < n)$),
- a function $d_i : X_n \to X_{n-1}$ (the i^{th} face map on n-simplices) ($n > 0$ and $0 \leq i \leq n$),
- for each surjective map $\sigma_i : [n+1] \to [n]$ of totally ordered sets,
- a function $s_i : X_n \to X_{n+1}$ (the i^{th} degeneracy map on n-simplices) ($n \geq 0$ and $0 \leq i \leq n$),
- such that these functions satisfy the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{when } i < j, \\ 1, & \text{when } i = j, j+1, \\ s_j d_{i-1}, & \text{when } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ when } i \leq j$$

The face maps, and degeneracy maps for the Nerve of a category are as follows:

- $d_i : N_k(\mathcal{C}) \to N_{k-1}(\mathcal{C})$:

$$\begin{array}{c}(A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots \to A_k) \\ \big\downarrow \\ (A_1 \to \cdots A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots A_k)\end{array}$$

- $s_i : N_k(\mathcal{C}) \to N_{k+1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_i \to \cdots \to A_k) \mapsto (A_1 \to \cdots A_i \xrightarrow{\text{id}} A_i \to \cdots A_k).$$

Some facts

- Examples of a model category which is not a Waldhausen category: Triangulated categories.
- The functor $\_ \otimes \_ : \Gamma(G_*) \times \Gamma(G_*) \to \Gamma(G_*)$ is in fact an oplax functor.

Oplax functor

If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that, for 1-cells f, g, we have $F(f \circ g) \cong F(f) \circ F(g)$ (but not exactly equal). Then the functor F is called as an oplax functor.

Suspension

Smash product

Let X, Y be two spaces. Then their smash product $X \wedge Y := X \times Y / X \vee Y$.

Example 8

$S^1 \wedge S^1 = S^2$, in fact $S^n \wedge S^m = S^{n+m}$ for any $n, m \in \mathbb{N}$.

Remark

- $\Sigma X \cong S^1 \wedge X$.
- $\Sigma^k X \cong S^k \wedge X$.

Remark

- In a category of R-modules, we have

$$\mathrm{Hom}(X \otimes A, Y) \cong \mathrm{Hom}(X, \mathrm{Hom(A,Y)}).$$

- Similarly, in case of pointed topological spaces, smash product plays the role of the tensor product. If A, X are compact Hausdorff then we have

$$\mathrm{Hom}(X \wedge A, Y) \cong \mathrm{Hom}(X, \mathrm{Hom(A,Y)}).$$

- So, in particular, for $A = S^1$, we have

$$\mathrm{Hom}(\Sigma X, Y) \cong \mathrm{Hom}(X, \mathrm{Hom}(S^1, Y)) = \mathrm{Hom}(X, \Omega Y).$$

- Here ΩY carries compact-open topology.
- This implies, the suspension functor $\Sigma \vdash \Omega$, the loop space functor.

Definition 9

Let X and Y be two topological spaces, and let $C(X,Y)$ denote the set of all continuous maps from X to Y. Given a compact subset K of X and an open subset U of Y, let $V(K,U)$ denote the set of all functions $f \in C(X,Y)$ such that $f(K) \subseteq U$. Then the collection of all such $V(K,U)$ is a subbase for the compact-open topology on $C(X,Y)$.

Properties of 2-CM

Proposition

Given a SQuad $G_0^{ab} \otimes G_0^{ab} \xrightarrow{w} G_1 \xrightarrow{\partial} G_0$. Then the homomorphism w is central.

Proof.

$$[a, w(\{y\} \otimes \{z\})] = w(\{\partial w(\{y\} \otimes \{z\})\} \otimes \{\partial(a)\})$$
$$= w(\{[z, y]\} \otimes \{\partial(a)\}) = w(0 \otimes \{\partial(a)\}) = 0.$$ □

Similar result is also true for SCM.

Proposition

Given a 2-CM G_*, $\pi_2(G_*)$ is abelian.

Proof.

From the result above, and the definition of $\pi_2(G_*)$, $\pi_2(G_*)$ is central in G_2, in particular it is abelian. □

Proposition

Given a 2-CM G_*, $\pi_1(G_*)$ is abelian.

Proof.

Im ∂ is normal in G_0 since:

$$\partial(f^x) = (\partial f)^x = x^{-1}(\partial f)x, f \in G_1, x \in G_0.$$

Similarly, $\pi_1(G_*)$ makes sense since $\mathrm{Im}(\partial : G_2 \to G_1)$ is normal in G_1, hence in particular in $\mathrm{Ker}(\partial : G_1 \to G_0)$. Then,

$f_0\partial\alpha_0 \cdot f_1\partial\alpha_1 = f_0f_1\partial(\alpha_0^{f_1}\alpha_1) = f_1f_2\partial(\langle f_0, f_1\rangle\alpha_0^{f_1}\alpha_1)$

$= f_1f_0\partial(\alpha_1^{f_0}\alpha_0)\partial((\alpha_1^{f_0}\alpha_0)^{-1}\langle f_0, f_1\rangle\alpha_0^{f_1}\alpha_1)$

$= f_1\partial\alpha_1 \cdot f_0\partial\alpha_0 \cdot \partial((\alpha_1^{f_0}\alpha_0)^{-1}\langle f_0, f_1\rangle\alpha_0^{f_1}\alpha_1)$ □

Spectrum

Definition 10

A spectrum $\mathbb{X}$ is a sequence

$$\cdots \to X_2 \to X_1 \to X_0.$$

of pointed spaces $\{X_n\}_{n\geq 0}$ with the structure maps $\Sigma X_n \to X_{n+1}$.

Definition 11

A stable quadratic module C_* is a commutative diagram of group homomorphisms To appendix

$$\begin{array}{ccc} C_0^{ab} \otimes C_0^{ab} & & \\ {\scriptstyle w}\downarrow & \overset{\text{commutator}}{\searrow} & \\ C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

such that given $c_i, d_i \in C_i, i = 0, 1,$

1. $w(\{\partial(c_1)\} \otimes \{\partial(d_1)\}) = [d_1, c_1] = d_1^{-1} c_1^{-1} d_1 c_1,$
2. $w(\{c_0\} \otimes \{d_0\} + \{d_0\} \otimes \{c_0\}) = 0.$ (The stability condition).

$$C_0 \to C_0^{ab}$$

$$x \mapsto \{x\}$$

Remark

The homotopy groups of C_* are:

- $\pi_0(C_*) = \mathrm{Coker}\partial,$
- $\pi_1(C_*) = \mathrm{Ker}\partial.$

Detailed SQuad structure for a Waldhausen category[2]

- The generators for dimension 0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for dimension 1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- such that the following relations hold (i.e., we define ∂, w):
  - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
  - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - $[0] = 0$.
  - $[A \xrightarrow{id} A] = 0$.
  - $[A \xrightarrow{id} A \twoheadrightarrow 0] = 0, [0 \rightarrowtail A \xrightarrow{id} A] = 0$.
  - For any composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

[2] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

- For any $A, B \in Ob(\mathcal{C})$, define the w as follows:

$$w([A] \otimes [B]) := \langle [A], [B] \rangle$$

$$=$$

$$-[\ B \overset{i_2}{\rightarrowtail} A \coprod B \overset{p_1}{\twoheadrightarrow} A\ ] + [\ A \overset{i_1}{\rightarrowtail} A \coprod B \overset{p_2}{\twoheadrightarrow} B\ ].$$

Here,

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \coprod B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are natural inclusions and projections of a coproduct in $\mathcal{C}$.

- For any commutative diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

we have

$$[A_0 \xrightarrow{\sim} A_1] + [B_0/A_0 \xrightarrow{\sim} B_1/A_1] + \langle [A], -[B_1/A_1] + [B_0/A_0] \rangle$$
$$=$$
$$-[A_1 \rightarrowtail B_1 \twoheadrightarrow B_1/A_1] + [B_0 \xrightarrow{\sim} B_1] + [A_0 \rightarrowtail B_0 \twoheadrightarrow B_0/A_0].$$

- For any commutative diagram consisting of cofiber sequences in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow\!\!\!\uparrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow\!\!\!\uparrow & & \uparrow\!\!\!\uparrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$

we have,

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

$$=$$

$$[\mathrm{A} \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle.$$