2-type of the K-theory of a Waldhausen category

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Outline

Waldhausen categories

Definition 1

Let $\mathcal{C}$ be a category equipped with a subcategory $co = co(\mathcal{C})$ of morphisms in the category $\mathcal{C}$ called cofibrations[a] (indicated with feathered arrows $\rightarrowtail$). The pair $(\mathcal{C}, co)$ is called a category with cofibrations if the following axioms are satisfied:

1. Every isomorphism in $\mathcal{C}$ is a cofibration.
2. There is a zero object, 0 in $\mathcal{C}$, and the unique morphism $0 \rightarrowtail A$ in $\mathcal{C}$ is a cofibration for every $A \in Ob(\mathcal{C})$. (i.e., every object of $\mathcal{C}$ is cofibrant).
3. If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \bigcup_A C$ of these two maps exists in $\mathcal{C}$ and $C \rightarrowtail B \bigcup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \bigcup_A C \end{array}$$

[a] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory.* American Mathematical Society, 2010, pp. 172–174.

Remarks

1. Coproduct $B \coprod C$ of any two objects $B, C \in Ob(\mathcal{C})$ exists.
   Since, $B \coprod C = B \bigcup_0 C$.
2. Every cofibration $A \rightarrowtail B$ in $\mathcal{C}$ has a cokernel B/A.
   Since, $B/A = B \bigcup_A 0$.
3. We refer to $A \rightarrowtail B \twoheadrightarrow B/A$ as a cofibration sequence in $\mathcal{C}$.

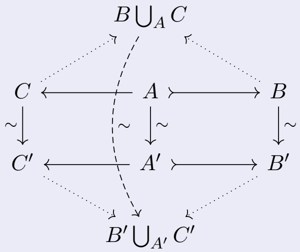
Example 2

1. The category $\mathbf{R}-\mathbf{Mod}$, for any ring R is a category with cofibrations:
   The cofibrations are the injective maps.
2. In fact, any exact category, hence any abelian category is naturally a category with cofibrations:
   The cofibrations are the monomorphisms.

Definition 3

A Waldhausen category $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called weak equivalences (abbreviated w.e. and indicated with $\xrightarrow{\sim}$) satisfying the following axioms:

1. Every isomorphism in $\mathcal{C}$ is a w.e.
2. Weak equivalences are closed under composition.
   (So we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$.)
3. Gluing axiom:



   The induced map $B \bigcup_A C \to B' \bigcup_{A'} C'$ is also a weak equivalence.

Definition 4

A Waldhausen category $\mathcal{C}$ is called saturated if whenever f, g are composable maps, and fg is a w.e., f is a w.e. if and only if g is.

Remark

We will consider only saturated Waldhausen categories, and hence we will just call them Waldhausen categories by abuse of language.

Example 5

The category of bounded above ($k \geq 0$) chain complexes over a ring R, $\mathbf{Ch}_R$ is a Waldhausen category by defining a map $f : M \to N \in Hom_{Ch_R}(M, N)$ is

- a w.e. if f induces isomorphism on homology groups.
- a cofibration if for each $k \geq 0$ the map $f_k : M_k \to N_k$ is a monomorphism with a projective module as its cokernel.

Example 6

Any category with cofibrations $(\mathcal{C}, co)$ may be considered as a Waldhausen category in which the category of weak equivalences is the category $\text{iso}(\mathcal{C})$ of all isomorphisms.

Definition 7

A functor between Waldhausen categories is exact if it is pointed ($0 \mapsto 0$), takes cofibrations to cofibrations, and w.e. to w.e., and preserves the pushout:

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \bigcup_A C \end{array}$$

K-theory of Waldhausen categories

Definition 8

Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$[a] is the abelian group presented as having one generator $[C]$ for each $C \in Ob(\mathcal{C})$, subject to following relations:

1. $[C] = [C']$ if there exists a w.e. $C \xrightarrow{\sim} C'$.
2. $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

[a] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory.* American Mathematical Society, 2010, pp. 172–174.

Remarks

These relations imply:

1. $[0] = 0$.
2. $[B \coprod C] = [B] + [C]$.
3. Since pushouts preserve cokernels, $[B \bigcup_A C] = [B] + [C] - [A]$.
4. $[B/A] = [B] - [A]$ since, $B/A = B \bigcup_A 0$.

- We will now see $S_\bullet$-construction. S stands for Segal as in Graeme B. Segal. Segal gave a similar construction for additive categories but it was reinvented by Waldhausen for Waldhausen categories.
- For any category $\mathcal{C}$, the arrow category[1] $Ar\mathcal{C}$ is the category with $Ob(Ar\mathcal{C})$ = Morphisms in $\mathcal{C}$, a morphism from $f : a \to b$ to $g : c \to d$ is a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccc} a & \longrightarrow & c \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ b & \longrightarrow & d \end{array}$$

- Consider $[n] = \{0 \leftarrow 1 \leftarrow \cdots \leftarrow n\}$ as a category, and the arrow category $Ar([n]^{op})$.
- For e.g. in $Ar([11]^{op})$ there is a unique morphism from the object $(2 \to 4)$ to $(3 \to 7)$ and no morphism in the other way.

[1] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The Local Structure of Algebraic K-Theory*. Springer-Verlag London, 2013, pp. 24–32.

Definition 9

Let $\mathcal{C}$ be a category with cofibrations. Then $S\mathcal{C} = \{[n] \mapsto S_n\mathcal{C}\}$ is the simplicial category which in degree n is the category $S_n\mathcal{C}$ of functors $C : Ar([n]^{op}) \to \mathcal{C}$ satisfying the following properties: To appendix

1. For all $j \geq 0$, $C(j = j) = 0$.
2. If $i \leq j \leq k$, then $C(i \leq j) \rightarrowtail C(i \leq k)$ is a cofibration, and

$$\begin{array}{ccc} C(i \leq j) & \longrightarrow & C(i \leq k) \\ \downarrow & & \downarrow \\ C(j = j) & \longrightarrow & C(j \leq k) \end{array}$$

   is a pushout.

- From the $S_\bullet$-construction, we can have for following:

$$S_\bullet w\mathcal{C} = \{[n] \mapsto Ob(S_n w\mathcal{C})\} \in \mathbf{sSet}.$$

So, we can have the loop space of the geometric realization:

$$K(\mathcal{C}) := \Omega|S_\bullet w\mathcal{C}|.$$

- Hence, we have:

$$\pi_i(K(\mathcal{C})) = \pi_i(\Omega|S_\bullet w\mathcal{C}|) \cong \pi_{i+1}(|S_\bullet w\mathcal{C}|) \overset{\text{def}}{=} \pi_{i+1}(S_\bullet w\mathcal{C}).$$

- We define a construction for a Waldhausen category $\mathcal{C}$, denoted by $T_\bullet\mathcal{C}$.
  Where, $T_n\mathcal{C}$ is generated by $N_p(S_q w\mathcal{C}),\ p+q-1=n$.
  Here, N stands for the nerve of the category, and w stands for considering weak equivalences.
- So, $N_p(S_q w\mathcal{C}) \in \mathbf{s^2Set}$. Up on taking its anti-diagonal (via a w.e. called Artin-Mazur map) becomes a **sSet**.

$$N_p S_q w\mathcal{C} \longmapsto d(N_p S_q w\mathcal{C}) \xmapsto{\text{Artin-Mazur}} T(N_p S_q w\mathcal{C})$$

- Since it is known that $Ob(S_\bullet w\mathcal{C}) \xrightarrow{\sim} d(N_p(S_q w\mathcal{C}))$, the two simplicial sets $Ob(S_\bullet w\mathcal{C})$ and $T_\bullet\mathcal{C}$ are weakly equivalent, so they have same homotopy groups.

Examples of cells

Example 10

Given a Waldhausen category $\mathcal{C}$, :
$T_0(\mathcal{C})$[a] consists of:

$$\text{A}$$

Figure 1: $N_0(S_1 w\mathcal{C})$

Similarly, for the 1-type:
$T_1(\mathcal{C})$ consists of:

$$A_0 \xrightarrow{\sim} A_1$$

Figure 2: $N_1(S_1 w\mathcal{C})$

$$A \rightarrowtail B \twoheadrightarrow C$$

Figure 3: $N_0(S_2 w\mathcal{C})$

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Example 11

Again, similarly, for the 2-type:
$T_2(\mathcal{C})$ consists of:

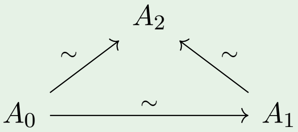


Figure 4: $N_2(S_1 w\mathcal{C})$

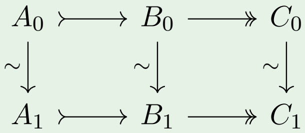


Figure 5: $N_1(S_2 w\mathcal{C})$

Example 11

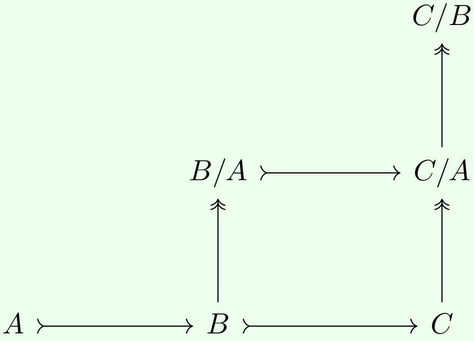


Figure 6: $N_0(S_3 w\mathcal{C})$

Motivation

Fact 12

- *Given a Waldhausen category $\mathcal{C}$, the simplicial set above is a zero-level of the spectrum $K(\mathcal{C})$. So, we get the following induced maps:*
- $$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{\quad biexact \quad} & \mathcal{E} \\ & \Big\downarrow & \\ K(\mathcal{C}) \wedge K(\mathcal{D}) & \longrightarrow & K(\mathcal{E}) \\ & \Big\downarrow & \\ \pi_i(K(\mathcal{C})) \times \pi_j(K(\mathcal{D})) & \longrightarrow & \pi_{i+j}(K(\mathcal{E})) \end{array}$$

Remark

- We know[a], for a given biexact functor between Waldhausen categories:

$$\mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

  we have the classical homomorphisms:
  - $K_0(\mathcal{C}) \times K_0(\mathcal{D}) \to K_0(\mathcal{E})$,
  - $K_0(\mathcal{C}) \times K_1(\mathcal{D}) \to K_1(\mathcal{E})$,
  - $K_1(\mathcal{C}) \times K_0(\mathcal{D}) \to K_1(\mathcal{E})$.
- So, extending this to 2-type, we expect to find the induced map:
  - $K_1(\mathcal{C}) \times K_1(\mathcal{D}) \to K_2(\mathcal{E})$.

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Approximation of a sSet by n-types

Definition 13

n-type[a] is the full subcategory of $\text{Top}^*/\cong$ (i.e., pointed topological spaces up to homotopy equivalence) consisting of connected CW-spaces Y with $\pi_i(Y) = 0$ for $i > n$. To appendix

[a] Hans-Joachim Baues. "Combinatorial Homotopy and 4-Dimensional Complexes". In: *Walter de Gruyter* (1991), pp. 171–177.

Fact 14

For a connected CW-complex X, one can construct a sequence of spaces P_nX such that $\pi_i(P_nX) \cong \pi_i(X)$ for $i \leq n$, and $\pi_i(P_nX) = 0$ for $i > n$, and for $i_n : X \to P_nX$, and $j_n : P_nX \to P_{n-1}X$ we have $j_n \circ i_n = i_{n-1}$ for all $n \geq 1$.

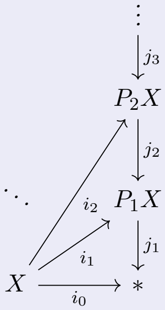


Figure 7: *Postnikov tower*

This commutative diagram is called a Postnikov tower[a] of X, the n-type spaces P_nX are called truncations of X.

[a] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002, pp. 10, 354–355.

Postnikov tower of a **sSet**

- If $X \in \mathbf{sSet}$, X is fibrant, then $P_n X = Cosk_n(X)$, the tower of Coskeletons[2] via Kan extensions.

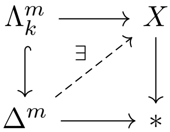


Figure 8: Fibrant object X in **sSet**

  - Λ^m_k is a horn.
  - The lift exists for each $m, k \in \mathbb{N}$, $k < m$.
- In general, if X is not fibrant, we can use a fibrant replacement $X \to RX$ where $P_n(X) = Cosk_n(RX)$.
- In general, the $\mathrm{S}_\bullet$-construction is not fibrant, so we work with a different (algebraic) model.

[2] W. G. Dwyer, D. M. Kan, and J. H. Smith. "An obstruction theory for simplicial categories". In: *Nederl. Akad. Wetensch. Indag. Math.* 48.2 (1986), pp. 153–161. ISSN: 0019-3577.

Models for n-types: $n = 0, 1, 2$

We want algebraic model for the types in the Postnikov tower:

$$
\begin{array}{ccc}
 & & P_2(T_*\mathcal{C}) \simeq N_*(D_*^{(2)}(\mathcal{C})) \\
 & \nearrow & \downarrow \\
 & & P_1(T_*\mathcal{C}) \simeq N_*(D_*^{(1)}(\mathcal{C})) \\
 & \nearrow & \downarrow \\
\mathcal{C} & \longrightarrow & P_0(T_*\mathcal{C})
\end{array}
$$

- $n = 0$: Group, a fundamental group.
- $n = 1$: $D_*^{(1)}(\mathcal{C})$: $D_1^{(1)}(\mathcal{C}) \xrightarrow{\partial} D_0^{(1)}(\mathcal{C})$, a SQuad.
- $n = 2$: $D_*^{(2)}(\mathcal{C})$: $D_2^{(2)}(\mathcal{C}) \xrightarrow{\partial} D_1^{(2)}(\mathcal{C}) \xrightarrow{\partial} D_0^{(2)}(\mathcal{C})$, a S2-CM.

Models for 1-type

- It is known that a stable quadratic module (SQuad)[3] is 1-type, so we construct a SQuad for the given Waldhausen category $\mathcal{C}$.
- Also, stable crossed modules (SCM) are models of (algebraic) 1-types.
- SQuad embeds as reflective subcategory in SCM.
  - A full subcategory $i : \mathcal{C} \to \mathcal{D}$ is reflective, if the inclusion functor i has a left adjoint.

[3] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

Definition 15

A stable quadratic module C_* is a commutative diagram of group homomorphisms To appendix

$$\begin{array}{ccc} C_0^{ab} \otimes C_0^{ab} & & \\ \downarrow w & \searrow^{\text{commutator}} & \\ C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

such that given $c_i, d_i \in C_i, i = 0, 1$,

1. $w(\{\partial(c_1)\} \otimes \{\partial(d_1)\}) = [d_1, c_1] = d_1^{-1} c_1^{-1} d_1 c_1$,
2. $w(\{c_0\} \otimes \{d_0\} + \{d_0\} \otimes \{c_0\}) = 0$. (The stability condition).

$$C_0 \to C_0^{ab}$$

$$x \mapsto \{x\}$$

Remark

The homotopy groups of C_* are:

- $\pi_0(C_*) = \text{Coker}\partial$,
- $\pi_1(C_*) = \text{Ker}\partial$.

1-type of a Waldhausen category

$$U : \mathbf{SQuad} \xrightarrow{\text{Forget}} \mathbf{Set} \times \mathbf{Set}$$

$$C_* \mapsto (C_0, C_1).$$

The functor U has a left adjoint F, and a SQuad $F(E_0, E_1)$ is called free stable quadratic module on the sets E_0 and E_1.

Fact 16

Given a Waldhausen category $\mathcal{C}$, we can define a corresponding SQuad $F(T_0(\mathcal{C}), T_1(\mathcal{C}))$[a], where $T_0(\mathcal{C})$, $T_1(\mathcal{C})$ come from example 10, 11. To appendix

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

A diagram of low types

$$
\begin{array}{ccccccc}
\text{SQuad} & \xhookrightarrow{\text{Reflect}} & \text{SCM} & \hookrightarrow & \text{S2-CM} & \longleftrightarrow & \text{SM 2-cat} \\
\downarrow{\scriptstyle \text{Forget}} & & \downarrow{\scriptstyle \text{Forget}} & & \downarrow{\scriptstyle \text{Forget}} & & \\
\text{Quad} & \hookrightarrow & \text{CM} & \hookrightarrow & \text{2-CM} & &
\end{array}
$$

- SQuad: Stable quadratic modules
- Quad: Quadratic modules
- CM: Crossed modules
- SCM: Stable Crossed modules
- 2-CM: 2-Crossed modules
- S2-CM: Stable 2-Crossed modules
- SM 2-Cat: Symmetric monoidal 2-Categories

Definition 17

A crossed module[a] G_* consists of a G_0-equivariant group homomorphism, where G_0 acts on itself by conjugation.

$$G_1 \xrightarrow{\partial} G_0$$

where the action of G_0 on G_1 satisfies

- $f^{\partial g} = g^{-1} f g$.

[a] H.-J. Baues and Daniel Conduché. "On the 2-type of an iterated loop space". In: *Forum Mathematicum* (1997), pp. 725–733.

Remark

The homotopy groups of the crossed module G_* are:

- $\pi_0(G_*) = \mathrm{Coker}\partial$,
- $\pi_1(G_*) = \mathrm{Ker}\partial$.

Definition 18

A 2-crossed module[a] G_* consists of a complex of G_0-groups

$$\begin{array}{ccccc} G_1 \times G_1 & & & & \\ \downarrow{\scriptstyle \langle\cdot,\cdot\rangle} & & & & \\ G_2 & \xrightarrow{\partial} & G_1 & \xrightarrow{\partial} & G_0 \end{array}$$

(so that $\partial\partial = 0$) and ∂'s are G_0-equivariant, where G_0 acts on itself by conjugation, such that $G_2 \xrightarrow{\partial} G_1$ is a crossed module such that

- $(\alpha^f)^x = (\alpha^x)^{f^x}$ for all $\alpha \in G_2, f \in G_1, x \in G_0$.
- There is a function $\langle\cdot,\cdot\rangle : G_1 \times G_1 \to G_2$ called Peiffer lifting satisfying:

  1. $\partial\langle f, g\rangle = f^{-1}g^{-1}fg^{\partial f}$,
  2. $\langle \partial\alpha, f\rangle = \alpha^{-1}\alpha^f$,
  3. $\langle f, \partial\alpha\rangle = (\alpha^{-1})^f \alpha^{\partial f}$,
  4. $\langle f, gh\rangle = \langle f, h\rangle\langle f, g\rangle^{h^{\partial f}}$,
  5. $\langle fg, h\rangle = \langle f, h\rangle^g \langle g, h^{\partial f}\rangle$,
  6. $\langle f, g\rangle^x = \langle f^x, g^x\rangle$.

  For all $x \in G_0, f, g, h \in G_1, \alpha \in G_2$. To appendix

[a] Ronald Brown and İlhan İçen. "Homotopies and Automorphisms of Crossed Modules of Groupoids". In: *Applied Categorical Structures* (2003),

Remark

The homotopy groups of a 2-crossed module G_* are:

- $\pi_0(G_*) = \mathrm{Coker}(\partial : G_1 \to G_0)$,
- $\pi_1(G_*) = \mathrm{Ker}(\partial : G_1 \to G_0)/(\mathrm{Im}(\partial : G_2 \to G_1))$,
- $\pi_2(G_*) = \mathrm{Ker}(\partial : G_2 \to G_1)$.

$$\begin{array}{ccccccc}
\text{SQuad} & \xrightarrow{\text{Reflect}} & \text{SCM} & \hookrightarrow & \text{S2-CM} & \longleftrightarrow & \text{SM 2-Cat} \\
\downarrow \text{Forget} & & \downarrow \text{Forget} & & \downarrow \text{Forget} & & \\
\text{Quad} & \hookrightarrow & \text{CM} & \hookrightarrow & \text{2-CM} & &
\end{array}$$

- S2-CM: Stable 2-Crossed modules
- SM 2-Cat: Symmetric monoidal 2-Categories

Why are we doing this?

- Why stability?: Because of the stability condition, the spectrum remains invariant under suspension, i.e., on taking suspension, the homotopy groups shift to next level without changing anything else.
- Why SM 2-Cat?: Because we know what SM 2-Cats are, whereas it is difficult to deduce the stabilization from 3-types.

SM 2-Cat structure on a 2-CM

Components of a Symmetric monoidal 2-Category[4] (SM 2-Cat are):

- A 2-Cat
- Monoidal structure ($\otimes$) on the 2-Cat
- Braiding (β) on the monoidal structure
- Left ($\eta_{-|--}$) and right ($\eta_{--|-}$) hexagonators
- Syllepsis (γ) (Exclusive for 2-Cat)

[4] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, 2021, pp. 384–396.

- Given a 2-CM G_*

$$G_2 \xrightarrow{\partial} G_1 \xrightarrow{\partial} G_0$$

We can have a 2-Cat $\Gamma(G_*)$:

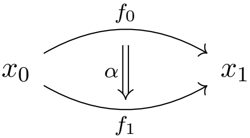
- $Ob(\Gamma(G_*)) = G_0$.

$$x_0 \in G_0.$$

- 1-Mor$(\Gamma(G_*)) = G_0 \times G_1$.

$$x_0 \xrightarrow{f_0} x_1 \text{ such that } x_1 = x_0\partial(f_0).$$

- 2-Mor$(\Gamma(G_*)) = G_0 \times G_1 \times G_2$.



Such that $f_1 = f_0\partial(\alpha)$.

Compositions of 2-cells:

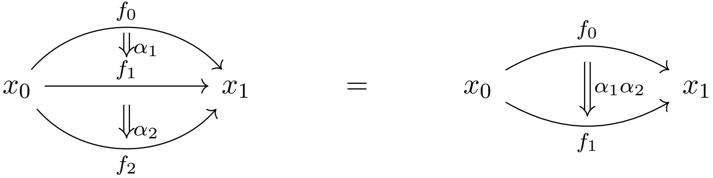


Figure 9: Vertical composition

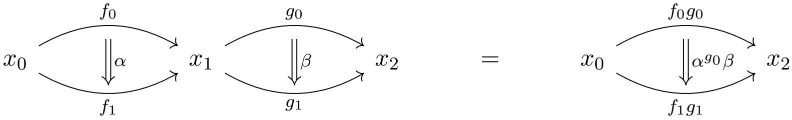


Figure 10: Horizontal composition

They satisfy certain compatibility conditions.

- Monoidal structure of 2-Crossed modules:

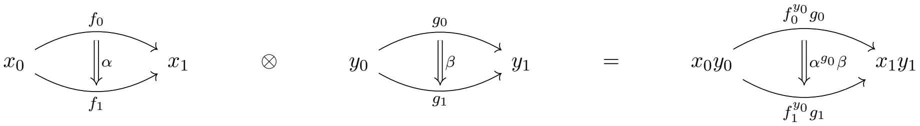


Figure 11: Monoidal structure

The functor $\_\otimes\_ : \Gamma(G_*) \times \Gamma(G_*) \to \Gamma(G_*)$ is in fact a lax functor.

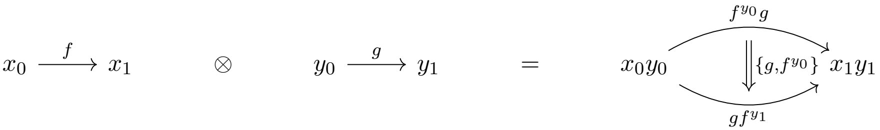
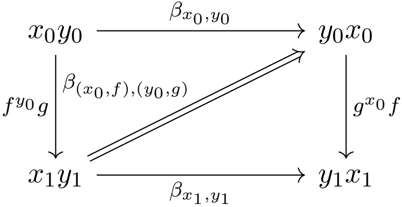


Figure 12: Lax functor

- Braiding:

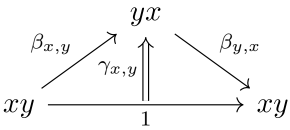
For every $x_0 \xrightarrow{f} x_1$ and $y_0 \xrightarrow{g} y_1$, we have



- Left ($\eta_{x|y,z}$) and right ($\eta_{x,y|z}$) hexagonators:

All these satisfy naturality and certain compatibility conditions.

- Syllepsis
  Given any two $x, y \in G_0$, we have



- For 1-Cat this 2-cell collapses to 1.
- They satisfy naturality and certain compatibility conditions.

References I

[1] Charles A. Weibel. *The K-book An Introduction to Algebraic K-theory*. American Mathematical Society, 2010, pp. 172–174.

[2] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The Local Structure of Algebraic K-Theory*. Springer-Verlag London, 2013, pp. 24–32.

[3] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

[4] Hans-Joachim Baues. "Combinatorial Homotopy and 4-Dimensional Complexes". In: *Walter de Gruyter* (1991), pp. 171–177.

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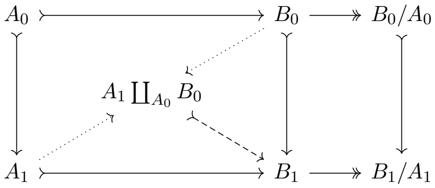
$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & S/K \\
 & & & & & & & & & & \twoheaduparrow \\
 & & & & & & & & K/N & \rightarrowtail & S/N \\
 & & & & & & & & \twoheaduparrow & & \twoheaduparrow \\
 & & & & & & N/A & \rightarrowtail & K/A & \rightarrowtail & S/A \\
 & & & & & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\
 & & & & A/H & \rightarrowtail & N/H & \rightarrowtail & K/H & \rightarrowtail & S/H \\
 & & & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\
 & & H/T & \rightarrowtail & A/T & \rightarrowtail & N/T & \rightarrowtail & K/T & \rightarrowtail & S/T \\
 & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow & & \twoheaduparrow \\
T & \rightarrowtail & H & \rightarrowtail & A & \rightarrowtail & N & \rightarrowtail & K & \rightarrowtail & S
\end{array}
$$

$S_2\mathcal{C}$ as a category with cofibrations

- Given a category with cofibrations $\mathcal{C}$, we can define a category called $S_2\mathcal{C}^{[2]}$ which has $(Ob(S_2\mathcal{C}))$ = collection of cofibration sequences, morphisms between two objects as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

- We can define cofibrations in the category $S_2\mathcal{C}$. A map like the one above is a cofibration if the vertical maps are cofibrations and the map from $A_1 \coprod_{A_0} B_0 \to B_1$ is a cofibration.



Remark

It can be seen that, with the similar pattern $S_n\mathcal{C}$ is a category with cofibrations for every $n \in \mathbb{N}$. Hence, one can consider $S_\bullet(S_\bullet\mathcal{C})$ and keep on doing this. This will give us a spectrum. However, we are not working with this spectrum in this study. We are just considering the first level of this spectrum, i.e., we are not considering the cofibration structure over $S_n\mathcal{C}$ for $n \geq 2$.

- Consider
$$U\colon \mathbf{SQuad} \xrightarrow{\text{Forget}} \mathbf{Set} \times \mathbf{Set}$$
$$C_* \mapsto (C_0, C_1).$$
The functor U has a left adjoint F, and a SQuad $F(E_0, E_1)$ is called free stable quadratic module[3] on the sets E_0 and E_1.
- Given a set E,
  - denote the free generated with basis E by $\langle E\rangle$,
  - free abelian group with basis E by $\langle E\rangle^{ab}$,
  - free group of nilpotency class 2 with basis E by $\langle E\rangle^{nil}$ (i.e., the quotient of $\langle E\rangle$ by triple commutators),
- Given an abelian group A,
  - denote the quotient of $A \otimes A$ by $a \otimes b + b \otimes a, a, b \in A$ by $\hat{\otimes}^2 A$.

- Given a pair of sets E_0 and E_1,
  - write $E_0 \cup \partial E_1$ for the set whose elements are the symbols e_0 and ∂e_1 for each $e_0 \in E_0, e_1 \in E_1$.

  Then we can define the free SQuad by considering:
  - $F(E_0, E_1)_0 = \langle E_0 \cup \partial E_1 \rangle^{nil}$,
  - $F(E_0, E_1)_1 = \hat{\otimes}^2 \langle E \rangle^{ab} \times \langle E_0 \times E_1 \rangle^{ab} \times \langle E_1 \rangle^{nil}$.

Simplicial Set

A simplicial set $X \in \mathbf{sSet}$ is

- for each $n \in \mathbb{N}$ a set $X_n \in \mathbf{Set}$ (the set of n-simplices),
- for each injective map $\partial_i : [n-1] \to [n]$ of totally ordered sets ($[n] := (0 < 1 < \cdots < n)$),
- a function $d_i : X_n \to X_{n-1}$ (the i^{th} face map on n-simplices) ($n > 0$ and $0 \le i \le n$),
- for each surjective map $\sigma_i : [n+1] \to [n]$ of totally ordered sets,
- a function $s_i : X_n \to X_{n+1}$ (the i^{th} degeneracy map on n-simplices) ($n \ge 0$ and $0 \le i \le n$),
- such that these functions satisfy the simplicial identities:

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & \text{when } i < j, \\ 1, & \text{when } i = j, j+1, \\ s_j d_{i-1}, & \text{when } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \text{ when } i \le j$$

Nerve of a category

Nerve of a small category $\mathcal{C}$ is a simplicial complex $N_\bullet(\mathcal{C})$.

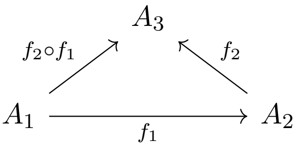
- $N_0(\mathcal{C})$ = 0-cells = $Ob(\mathcal{C})$:

$$\bullet A$$

- $N_1(\mathcal{C})$ = 1-cells = Morphisms of $\mathcal{C}$:

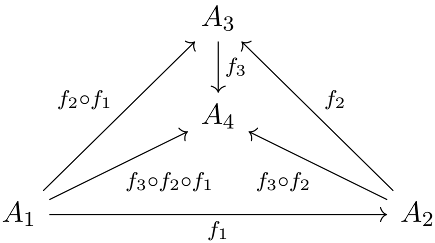
$$A_1 \xrightarrow{f} A_2$$

- $N_2(\mathcal{C})$ = 2-cells = A pair of composable morphisms in $\mathcal{C}$:



i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$.

- $N_3(\mathcal{C})$ = 3-cells = A triplet of composable morphisms in $\mathcal{C}$:



  i.e., generated from $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$.
- and so on.
- $d_i : N_k(\mathcal{C}) \to N_{k-1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots \to A_k)$$
$$\downarrow$$
$$(A_1 \to \cdots A_{i-1} \xrightarrow{f_i \circ f_{i-1}} A_{i+1} \to \cdots A_k)$$

- $s_i : N_k(\mathcal{C}) \to N_{k+1}(\mathcal{C})$:

$$(A_1 \to \cdots \to A_i \to \cdots \to A_k) \mapsto (A_1 \to \cdots A_i \xrightarrow{\text{id}} A_i \to \cdots A_k).$$

Definition of a Quad[5]

Definition 19

A pre-crossed module G_* is a equivariant G_0-group homomorphism $\partial : G_1 \to G_0$, where G_0 acts on itself by conjugation.

[5] Hans-Joachim Baues. "Combinatorial Homotopy and 4-Dimensional Complexes". In: *Walter de Gruyter* (1991), pp. 171–177.

Definition 20

A quadratic module (w, δ, ∂) is a complex of G_0-groups

$$\begin{array}{ccccc} & & (G_1^{\mathrm{cr}})^{\mathrm{ab}} \times (G_1^{\mathrm{cr}})^{\mathrm{ab}} & & \\ & \swarrow^{w} & \downarrow \mathrm{w} & & \\ G_2 & \xrightarrow{\delta} & G_1 & \xrightarrow{\partial} & G_0 \end{array}$$

where, G_1^{cr} is a group such that the pre-cross module $\partial : G_1 \to G_0$ becomes a crossed module $\partial : G_1^{\mathrm{cr}} \to G_0$.

such that

- $\partial : G_1 \to G_0$ is a $nil(2)$-module.
- $\partial\delta = 0$, $\delta w = \mathrm{w} =$ Peiffer commutator map:
$$\mathrm{w}(x \otimes y) = -x - y + x + y^{\partial x}$$
- All homomorphisms are equivariant with respect to the action of G_0
- $f^{\partial x} = f + w(\{\partial f\} \otimes \{x\} + \{x\} + \{\partial f\})$ for all $f \in G_2, x \in G_1$.
- $w(\{\partial a\} \otimes \{\partial b\}) = [a, b] = -a - b + a + b$.

Remark

- Putting $G_0 = 0$ in the definition above gives us the Definition 15.
- Homotopy groups of the quadratic module $\sigma = (w, \delta, \partial)$ can be defined as:
  - $\pi_1(\sigma) = \mathrm{Coker}(\partial)$,
  - $\pi_2(\sigma) = \mathrm{Ker}(\partial)/\mathrm{Im}(\delta)$,
  - $\pi_3(\sigma) = \mathrm{Ker}(\delta)$.
- From Definition 15, we can conclude that C_0 and C_1 are groups of nilpotency class 2.
  - Given $x, y, z \in C_0$, we have:

    $$[x, [y, z]] = \partial w(\{[y, z]\} \otimes \{x\}) = \partial w(0 \otimes \{x\}) = 0.$$

  - Similarly, given $f, g, h \in C_1$ we have:
    $[f, [g, h]] = w(\{\partial([g, h])\} \otimes \{\partial(f)\}) = w(\{[\partial(g), \partial(h)]\} \otimes \{\partial(f)\}) = w(0 \otimes \{\partial(f)\}) = 0.$

Detailed SQuad structure for Fact 16[6]

- The generators for dimension 0 are:
  - $[A]$ for any $A \in Ob(\mathcal{C})$.
- The generators for dimension 1 are:
  - $[A_0 \xrightarrow{\sim} A_1]$ for any w.e.
  - $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence.
- such that the following relations hold (i.e., we define ∂, w):
  - $\partial([A_0 \xrightarrow{\sim} A_1]) = -[A_1] + [A_0]$.
  - $\partial([A \rightarrowtail B \twoheadrightarrow B/A]) = -[B] + [B/A] + [A]$.
  - $[0] = 0$.
  - $[A \xrightarrow{id} A] = 0$.
  - $[A \xrightarrow{id} A \twoheadrightarrow 0] = 0, [0 \rightarrowtail A \xrightarrow{id} A] = 0$.
  - For any composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

[6] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

- For any $A, B \in Ob(\mathcal{C})$, define the w as follows:

$$w([A] \otimes [B]) := \langle [A], [B] \rangle$$
$$=$$
$$-[\ B \overset{i_2}{\rightarrowtail} A \coprod B \overset{p_1}{\twoheadrightarrow} A\ ] + [\ A \overset{i_1}{\rightarrowtail} A \coprod B \overset{p_2}{\twoheadrightarrow} B\ ].$$

Here,

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \coprod B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are natural inclusions and projections of a coproduct in $\mathcal{C}$.

- For any commutative diagram in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} A_0 & \rightarrowtail & B_0 & \twoheadrightarrow & B_0/A_0 \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & B_1/A_1 \end{array}$$

we have

$$[A_0 \xrightarrow{\sim} A_1] + [B_0/A_0 \xrightarrow{\sim} B_1/A_1] + \langle [A], -[B_1/A_1] + [B_0/A_0] \rangle$$
$$=$$
$$-[A_1 \rightarrowtail B_1 \twoheadrightarrow B_1/A_1] + [B_0 \xrightarrow{\sim} B_1] + [A_0 \rightarrowtail B_0 \twoheadrightarrow B_0/A_0].$$

- For any commutative diagram consisting of cofiber sequences in $\mathcal{C}$ as follows:

$$\begin{array}{ccccc} & & & & C/B \\ & & & & \uparrow\!\!\!\!\twoheadrightarrow \\ & & B/A & \rightarrowtail & C/A \\ & & \uparrow\!\!\!\!\twoheadrightarrow & & \uparrow\!\!\!\!\twoheadrightarrow \\ A & \rightarrowtail & B & \rightarrowtail & C \end{array}$$

we have,

$$[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A]$$

$$=$$

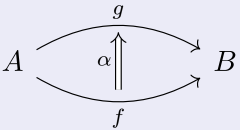
$$[\mathrm{A} \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B] + \langle [A], -[C/A] + [C/B] + [B/A] \rangle.$$

Definition of 2-categories

Definition 21

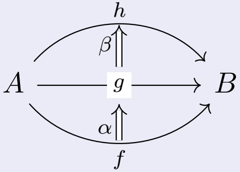
A (strict) 2-category $\mathcal{C}$ is comprised of the following: Back to main

- 0-Cells (Objects): Denoted by $Ob(\mathcal{C})$.
- 1-Cells (Morphisms): For $A, B \in Ob(\mathcal{C})$, a set $\mathrm{Hom}(A, B)$ of 1-cells from A to B, also known as morphisms. A 1-cell is often written textually as $f : A \to B$ or graphically as $A \xrightarrow{f} B$.
- 2-Cells: For $A, B \in Ob(\mathcal{C})$, $f, g \in \mathrm{Hom}(A, B)$, a set $\mathrm{Face}(f, g)$ of 2-cells from f to g. A 2-cell is often written textually as $\alpha : f \Rightarrow g : A \to B$ or graphically as follows:

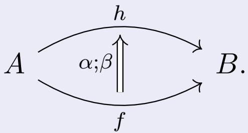


Definition 21

- 1-Identities: For each $A \in Ob(\mathcal{C})$, a 1-cell $A \xrightarrow{id_A} A$.
- 1-Composition: For each chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, a 1-cell $A \xrightarrow{f;g} C$.
- Vertical 2-Composition: For a chain of 2-cells

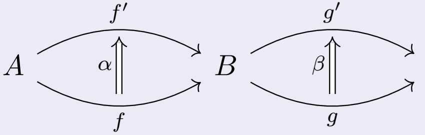


, a 2-cell

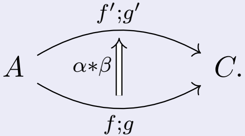


Definition 21

- Horizontal 2-Composition: For each chain of 2-cells



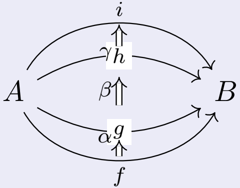
, a 2-cell



- 1-Identity: Every 1-cell $A \xrightarrow{f} B$ satisfies $(id_A; f) = f = (f; id_B)$.
- 1-Associativity: Every chain of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ satisfies $(f; g); h = f; (g; h)$.
- Vertical 2-Identity: Every 2-cell $\alpha : f \Rightarrow g : A \to B$ satisfies $id_f; \alpha = \alpha = \alpha; id_g$.

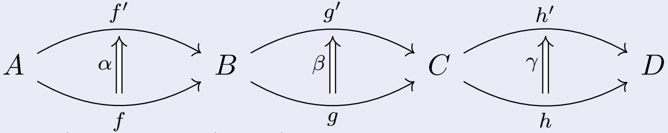
Definition 21

- Vertical 2-Associativity: Every chain of 2-cells



satisfies $(\alpha;\beta);\gamma = \alpha;(\beta;\gamma)$.

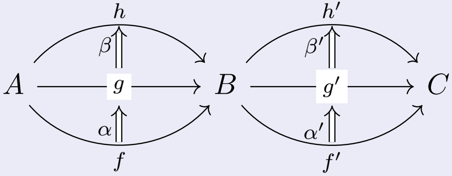
- Horizontal 2-Identity: Every 2-cell $\alpha : f \Rightarrow g : A \to B$ satisfies $id_{id_A} * \alpha = \alpha = \alpha * id_{id_B}$.
- 2-Identity: Every sequence of 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$ satisfies $id_f * id_g = id_{f;g}$.
- Horizontal 2-Associativity: Every chain of 2-cells



satisfies $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$.

Definition 21

- 2-Interchange: Every clover of 2-cells



satisfies

$(\alpha;\beta) * (\alpha';\beta') = (\alpha * \alpha');(\beta * \beta')$.

SQuad embeds as a reflective subcategory of SCM

Definition 22

A stable crossed module (SCM)[a] G_* is a crossed module $\partial : G_1 \to G_0$ together with a map Back to main

$$\langle \cdot, \cdot \rangle : G_0 \times G_0 \to G_1$$

satisfying the following for any $f, g \in G_1, x, y, z \in G_0$:

1. $\partial\langle x, y\rangle = [y, x]$,
2. $f^x = f + \langle x, \partial(f)\rangle$,
3. $\langle x, y + z\rangle = \langle x, y\rangle^z + \langle x, z\rangle$,
4. $\langle x, y\rangle + \langle y, x\rangle = 0$.

---

[a] Fernando Muro and Andrew Tonks. "The 1-type of a Waldhausen K-theory spectrum". In: *Advances in Mathematics 216* (2007), pp. 179–183.

5 In SCM, the condition in the definition of crossed modules is now equivalent to $\langle \partial(f), \partial(g)\rangle = [g, f]$.

Proposition

The category SQuad is a full subcategory of the category SCM given by those objects

$$C_0 \times C_0 \xrightarrow{\langle \cdot , \cdot \rangle} C_1 \xrightarrow{\partial} C_0$$

which satisfy

$$\langle c, [c', c''] \rangle = 0, \text{ for each } c, c', c'' \in C_0. \tag{1}$$

Proof.

We will first prove that SQuad embeds as a full subcategory of SCM. We claim that a SQuad C_* yields a SCM:

$$C_0 \times C_0 \xrightarrow{(c,d) \mapsto \{c\} \otimes \{d\}} C_0^{ab} \otimes C_0^{ab} \xrightarrow{w} C_1 \xrightarrow{\partial} C_0, \qquad C_0 \times C_0 \xrightarrow{\langle \cdot , \cdot \rangle} C_1$$

Moreover,

$$\langle c, [c', c''] \rangle = w(\{c\} \otimes \{[c', c'']\}) = w(\{c\} \otimes 0) = 0.$$

$\square$

Proof.

Axioms (1), (4), (5) in Definition 22 follow immediately from the Definition 15, and axiom (3) is consequence of the following:

$\langle c, c' + c'' \rangle = w(\{c\} \otimes (\{c'\} + \{c''\})) = w(\{c\} \otimes \{c'\}) + w(\{c\} \otimes \{c''\})$
$= \langle c, c' \rangle + \langle c, c'' \rangle = \langle c, c' \rangle + \langle c'', [c', c] \rangle + \langle c, c'' \rangle$ by (1)
$= \langle c, c' \rangle + \langle c'', \partial \langle c', c \rangle \rangle + \langle c, c'' \rangle$ by axiom (1) in Definition 22
$= \langle c, c' \rangle^{c''} + \langle c, c'' \rangle$ by axiom (2) in Definition 22.

Conversely, let us see that a SCM satisfying (1) can be obtained from a SQuad. Indeed, (1) and Definition 22 (4) imply that $\langle \cdot, \cdot \rangle$ factors through $C_0^{ab} \times C_0^{ab}$. Moreover, by (1), and Definition 22 (1), (2) the elements of C_0 act trivially on the image of $\langle \cdot, \cdot \rangle$, therefore $\langle \cdot, \cdot \rangle$ is bilinear by (3) in Definition 22. So $\langle \cdot, \cdot \rangle$ factors through $C_0^{ab} \otimes C_0^{ab}$ to give us the required SQuad. Remaining details and a proof of the reflective part can be read in [3]. □

Properties of 2-CM

Proposition

Given a SQuad $G_0^{ab} \otimes G_0^{ab} \xrightarrow{w} G_1 \xrightarrow{\partial} G_0$. Then the homomorphism w is central. Back to main

Proof.

$[a, w(\{y\} \otimes \{z\})] = w(\{\partial w(\{y\} \otimes \{z\})\} \otimes \{\partial(a)\})$
$= w(\{[z, y]\} \otimes \{\partial(a)\}) = w(0 \otimes \{\partial(a)\}) = 0.$ □

Similar result is also true for SCM.

Proposition

Given a 2-CM G_*, $\pi_2(G_*)$ is abelian.

Proof.

From the result above, and the definition of $\pi_2(G_*)$, $\pi_2(G_*)$ is central in G_2, in particular it is abelian. □

Proposition

Given a 2-CM G_*, $\pi_1(G_*)$ is abelian.

Proof.

Im ∂ is normal in G_0 since:

$$\partial(f^x) = (\partial f)^x = x^{-1}(\partial f)x, f \in G_1, x \in G_0.$$

Similarly, $\pi_1(G_*)$ makes sense since $\mathrm{Im}(\partial : G_2 \to G_1)$ is normal in G_1, hence in particular in $\mathrm{Ker}(\partial : G_1 \to G_0)$. Then,

$f_0\partial\alpha_0 \cdot f_1\partial\alpha_1 = f_0 f_1 \partial(\alpha_0^{f_1}\alpha_1) = f_1 f_2 \partial(\langle f_0, f_1\rangle \alpha_0^{f_1}\alpha_1)$

$= f_1 f_0 \partial(\alpha_1^{f_0}\alpha_0)\partial((\alpha_1^{f_0}\alpha_0)^{-1}\langle f_0, f_1\rangle \alpha_0^{f_1}\alpha_1)$

$= f_1\partial\alpha_1 \cdot f_0\partial\alpha_0 \cdot \partial((\alpha_1^{f_0}\alpha_0)^{-1}\langle f_0, f_1\rangle \alpha_0^{f_1}\alpha_1)$ □

Proposition

Given a 2-CM G_*, $\pi_2(G_*)$ is a $\pi_0(G_*)$-module.

Proof.

The homotopy group $\pi_2(G_*)$ is a subset of G_2, and $\pi_1(G_*)$ is a subset of G_0, so we can consider the multiplication $\alpha \cdot x = \alpha^x$. So, the only thing to check is $\mathrm{Im}(\partial : G_1 \to G_0)$ acts trivially on $\mathrm{Ker}(\partial : G_2 \to G_1)$.
Let $\alpha \in \mathrm{Ker}(\partial : G_2 \to G_1)$, $x = \partial f$ for some $f \in G_1$, then:
$\alpha^{\partial f} = \alpha^f \langle f, \partial\alpha \rangle = \alpha^f \langle f, 1 \rangle = \alpha^f$ (assuming $\langle f, 1 \rangle = 1$)
$= \alpha \langle \partial\alpha, f \rangle = \alpha \langle 1, f \rangle = \alpha$ (assuming $\langle 1, f \rangle = 1$). □

Coskeletons as a Postnikov decomposition[7]

- Given any $X \in \mathbf{sSet}$, we can have a truncation functor for each $n \in \mathbb{N}$
$$tr_n : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}.$$

- Then by Kan extension we have the following functors:
$$\mathbf{sSet} \;\underset{cosk_n}{\overset{\overset{sk_n}{\longleftarrow}}{\underset{\longleftarrow}{\xrightarrow{tr_n}}}}\; \mathbf{sSet}_{\leq n}$$
such that $sk_n \dashv tr_n \dashv cosk_n$.

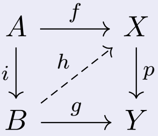
- Now consider,
$$Sk_n := sk_n \circ tr_n : \mathbf{sSet} \to \mathbf{sSet},$$
$$Cosk_n := cosk_n \circ tr_n : \mathbf{sSet} \to \mathbf{sSet}.$$
Then $Sk_n \dashv Cosk_n$.

[7] W. G. Dwyer, D. M. Kan, and J. H. Smith. “An obstruction theory for simplicial categories”. In: *Nederl. Akad. Wetensch. Indag. Math.* 48.2 (1986), pp. 153–161. ISSN: 0019-3577.

- They also satisfy the following properties:
  - $(Cosk_n X)_k \cong sSet(\Delta^k, Cosk_n X) \cong sSet(Sk_n \Delta^k, X)$.
  - If $k \leq n$: $Sk_n \Delta^k = \Delta^k$, $(Cosk_n X)_k = X_k$.
  - If $k = n+1$:
    $(Cosk_n X)_{n+1} \cong sSet(Sk_n \Delta^{n+1}, X) \cong sSet(\partial \Delta^{n+1}, X) = 0$.
- $Cosk_n$ is a right adjoint, so it preserves fibrant object. So, when X is fibrant, then so is $Cosk_n X$ and its homotopy groups are trivial in dimension $\geq n$.
- Hence, the sequence:
  $\mathrm{X} = \varprojlim \big(\cdots \to Cosk_{n+1}(X) \to Cosk_n(X) \to Cosk_{n-1}(X) \to \cdots \to * \big)$
  is up to homotopy, a Postnikov decomposition of X.

Definition 23

A map $i : A \to B$ is said to have the left lifting property (LLP)[a] with respect to another map $p : X \to Y$ and p is said to have the right lifting property (RLP) with respect to i if a lift $h : B \to X$ exists for any of the commutative diagram of the following form: Back to main



[a] W. G. Dwyer and J. Spaliński. "Homotopy theories and model categories". In: *Handbook of algebraic topology.* North-Holland, Amsterdam, 1995, pp. 73–126. DOI: 10.1016/B978-044481779-2/50003-1. URL: https://doi.org/10.1016/B978-044481779-2/50003-1.

Fact 24

The fibrations (in sense of Model category) are the maps which have the RLP with respect to acyclic cofibrations (i.e., cofibrations that are also w.e.).

Definition 25

An object A is called fibrant, if $A \to 0$ is a fibration.