Vector Bundles from an Algebraic Viewpoint

A thesis submitted in partial fulfillment of the requirements for the award of the degree of

BS-MS Dual Degree
by

Milind Vishwas Gunjal
14MS064

under the supervision of

Dr. Somnath Basu

to the

Department of Mathematics and Statistics

IISER KOLKATA

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH KOLKATA

May 2019

Declaration

I hereby declare that this thesis is my own work and to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Kolkata or any other educational institution, except where due acknowledgement is made in the thesis.

May 2019
IISER Kolkata

Milind Vishwas Gunjal

Certificate

It is certified that the research work included in the thesis report entitled *"Vector Bundles from an Algebraic Viewpoint"* has been carried out by Mr. Milind Vishwas Gunjal (14MS064) under my supervision and guidance to be submitted to partially fulfill the requirements for awarding the Master of Science degree by Indian Institute of Science Education and Research, Kolkata. The content of this project report has not been submitted elsewhere for the award of any academic and professional degree.

May 2019 — Somnath Basu
IISER Kolkata

Abstract

In this thesis, we will give a brief introduction to projective modules and vector bundles and discuss the correspondence between modules over the ring of continuous functions ($C(X)$) and vector bundles over X. We will also introduce covering space theory to get a better understanding of correspondence between algebraic and topological objects. A detailed discussion about the compactifications is also presented. We will also establish the notion of localization of vector bundles in terms of localization of $C(X)$-modules at a point using Swan's theorem. We will discuss some examples of Swan's theorem. Finally, we motivate the statement of the Quillen-Suslin theorem.

Keywords: Vector bundles, double covers, finitely generated projective $C(X)$-modules, localization, compactification.

Acknowledgements

Firstly, I would like to express my sincere gratitude to my advisor Dr. Somnath Basu for the continuous support of my BS-MS study and related research, for his patience, motivation, and immense knowledge. The door to Somnath sir's office was always open whenever I ran into a trouble spot or had a question about my research or writing. He consistently allowed this thesis to be my own work, but steered me in the right the direction whenever he thought I needed it. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my MS study. I would also like to thank all my teachers who were always keen to help me whenever I required one.

My sincere thanks also goes to Dr. Priyavrat Deshpande who provided me an opportunity to join as intern. Without his precious support and motivation it would not be possible to conduct this study.

I thank Jeevan Jankar, Rajat Mishra, Sachchidanand bhaiyya, Prahlad da for the stimulating discussions, for the sleepless nights we were working together before deadlines.

With a special mention to Mithu Mondal, Sribas Malik, Surya Ghosh, Dhanashree Somani, Indrajit, Arunima and my whole volleyball team in general, it was fantastic to have the opportunity to spend time with you. I thank you for all the fun we have had in the last five years. I would specially like to thank Anoop cheta, I am what I am because of him.

Last but not the least, I would like to thank my family: my parents and my sister for providing me with unfailing support and continuous encouragement throughout my years of study and my life in general.

Contents

List of Figures

Symbols

$\mathbb{N}$ Set of natural numbers

$\mathbb{Z}$ Set of integers

$\mathbb{Q}$ Set of rational numbers

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^n$ n dimensional real Euclidean space

$\mathbb{k}$ Field

$\mathbb{S}^n$ Unit sphere of dimension n

$\mathbb{RP}^n$ n dimensional real projective space

$M_n(\mathbb{R})$ Set of $n \times n$ matrices with real entries

$O_k(\mathbb{R})$ Set of $k \times k$ orthogonal matrices with real entries

A^t Transpose of matrix A

$dim_K R$ Dimension of R with respect to K

Id, id Identity map

$\langle a \rangle$ Ideal generated by an element a

$A - B$ Set of all points contained in A but not in B

A^C Complement of set A

$\mathbb{P}(X)$ Set of all subsets of the set X

A/B Quotient of A modulo B as groups, rings, modules, topological spaces, vector spaces

$\overline{A}$ Closure of set A

$f|_V$ Function f restricted to set V

$f \circ g$ Function f composed with the function g

$ker(f)$ Kernel of morphism f

$im(f)$ Image of map f

pr_i	Projection map onto i^{th} coordinate
$\overline{\gamma}$	Path with the reverse orientation as that of γ
$\varnothing$	Empty set
$\oplus, \otimes$	Direct sum, tensor product
$\bigsqcup$	Disjoint union
$\times, \prod$	Product of sets, groups, spaces or maps
$\cong$	Isomorphism/Homeomorphism

Preface

The purpose of this thesis is to study vector bundles and their correspondences with algebraic objects like projective modules over ring of continuous functions. The aim of the study was to reach to Quillen-Suslin theorem from a natural path of motivations. This theorem was also known as Serre's conjecture. Serre made some progress towards a solution in 1957 when he proved that every finitely generated projective module over a polynomial ring over a field was stably free, meaning that after forming its direct sum with a finitely generated free module, it becomes free. The problem remained open until 1976, when Daniel Quillen and Andrei Suslin independently proved that the answer was affirmative. Quillen was awarded the Fields Medal in 1978 in part for his proof of the Serre conjecture. Leonid Vaserstein later gave a simpler and much shorter proof of the theorem which can be found in [11].

In this thesis we will start by studying some basic properties of the ring of continuous functions in chapter 1 and some initial part of chapter 2. One can refer [17] for the same. Later in the second chapter we study Hilbert's Nullstellensatz and try to define a homeomorphism between the space X and $maxspec(C(X))$ with the Zariski topology when X is compact and Hausdorff. However, some details are mentioned in appendix B. Later I studied Tychonoff spaces and their compactifications in the hope of extending this homeomorphism over these spaces. This part, however, is included as appendix C in the thesis.

In chapter 3 we start with some basic concepts of covering space theory and study the very famous Galois correspondence from [9], which is between subgroups of fundamental group and covering spaces. Later in this chapter, we study some basic properties of vector bundles and give a correspondence between double covers and line bundles. However, one can study it more generally using principal G-bundles. In chapter 4, we introduce the notion of sections of vector bundles. Later, using the properties studied in initial chapters and otherwise, we study a correspondence between vector bundles and finitely generated projective $C(X)$-modules when X is compact and Hausdorff. Then in chapter 5, we start with some basic concepts of localization in algebraic sense, and try to correlate them with those in geometric sense, like, restricting a vector bundle to a trivializing neighbourhood. Taking motivation from this, we prove that finitely generated projective modules over local rings are free using Nakayama's lemma. We further generalize this result by relaxing the condition of the module being finitely generated. This is called Kaplansky's theorem. Seeing the local correspondence between these structures, it motivates us to ask if the same happen globally? We see that bundles over Euclidean spaces are trivializable and the corresponding projective module of sections are free and also, we have a result that freeness of projective modules is satisfied in the algebraic category of vector bundle over affine spaces, that is, where all the functions involved are not just continuous functions, but are polynomials, which is a version of the Quillen-Suslin theorem.

We will assume that the reader has some exposure to basic Algebraic and Differential Topology. We will try to give most of the proofs of the results that we state in this thesis, but sometimes due to some technical difficulties, we will skip some proofs and give appropriate reference for the readers.

In spite of best efforts of the author, there might be some errors of both typographical and mathematical in nature. The author is solely responsible for such errors.

1 Introduction

In this section we are setting up a basic machinery on the ring of continuous functions. We will refer [17] for the the whole section.

1.1 General topology

Given a topological space X, let $C(X)$ denote the set of continuous maps from X to $\mathbb{R}$. Then we have the following results:

Proposition 1.1. *$C(X)$ is a commutative ring with unity.*

Proof. We can define $(f+g)(x) := f(x) + g(x)$ for all $x \in X$, for all $f, g \in C(X)$.
Claim 1: $(f+g) \in C(X)$ for all $f, g \in C(X)$.
Let us consider the following functions:

$$\psi_1 : X \to X \times X, x \mapsto (x, x) \text{ for all } x \in X,$$

$$\psi_2 : X \times X \to \mathbb{R} \times \mathbb{R}, (x, y) \mapsto (f(x), g(y)) \text{ for all } (x, y) \in X \times X,$$

$$\psi_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto x + y \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

So, it is enough to show that ψ_1, ψ_2, ψ_3 are continuous, individually. This follows from the definition of continuity, topology of product space for ψ_1 and continuity of f, g implies that ψ_2 is continuous. Now, for ψ_3 consider an open interval (a, b) in $\mathbb{R}$ (since, open intervals are basis elements of $\mathbb{R}$). Consider $\psi_3^{-1}((a, b))$ this looks like an open strip whose boundary passes through $(0, a), (a, 0), (b, 0), (0, b)$. To show that this is an open set in $\mathbb{R}^2$ let us consider a point (x_0, y_0)in this strip. Then, the open ball of radius

$$\varepsilon_{(x_0,y_0)} = min\left\{\frac{|x_0 + y_0 - a|}{\sqrt{2}}, \frac{|x_0 + y_0 - b|}{\sqrt{2}}\right\}$$

centered around (x_0, y_0) lies totally inside the open strip. Hence claim 1 is proved.
Define $(f+g)(x) := f(x) + g(x)$ for all $x \in X$, for all $f, g \in C(X)$.
Claim 2: If we define $(f \cdot g)(x) := f(x) \cdot g(x)$ for all $x \in X$, then $(f \cdot g) \in C(X)$ for all $f, g \in C(X)$.
The proof for this goes like the previous one except, $\psi_3(x, y) = x \cdot y$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. So, again consider $(a, b) \subseteq \mathbb{R}$ then,

$$\psi_3^{-1}((a, b)) = \begin{cases} \text{region 1,} & \text{if } a > 0 \\ \text{region 2,} & \text{if } a \leq 0, b \geq 0 \\ \text{region 3,} & \text{if } b < 0 \end{cases}$$

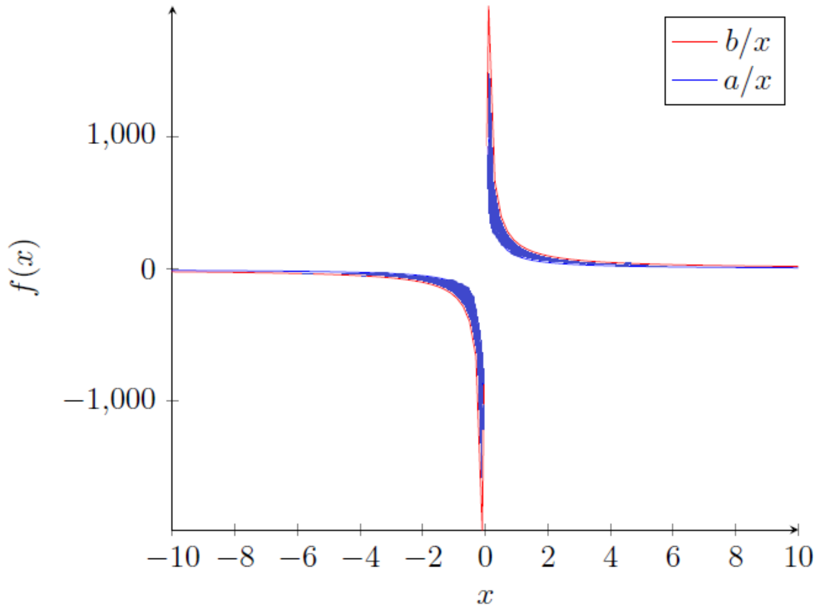


Figure 1: Region 1.

Note that these regions do not contain their respective boundaries. We need to show that these three regions are open subsets of $\mathbb{R}^2$. Let us define $d_c\,(x_0, y_0) :=$ distance between the point (x_0, y_0) and the curve xy = c. Note that two perpendiculars can be drawn from any point to the hyperbola. We will consider the smaller length to be the defined distance. So, for every (x_0, y_0) in every region we can obtain an open ball of radius $\varepsilon_{(x_0,y_0)} = min\{d_a(x_0, y_0), d_b(x_0, y_0)\}$, centered at (x_0, y_0) which will lie completely inside the respective regions. Hence, claim 2 is proved. Now, since the addition and multiplication is defined, one can easily verify that associativity under addition, commutativity under addition, distributivity of multiplication over addition follows directly from the ring structure of $\mathbb{R}$. Also, one can observe that the constant zero function acts as the additive identity and $(-f)(x) := -(f(x))$ for all $x \in X$ acts as the additive inverse. Hence, this gives the required the ring structure of $C(X)$.
We see that the commutativity of multiplication in $\mathbb{R}$ induces commutativity of multiplication in $C(X)$ as well. Also, note that the constant 1 function acts as unity of this ring. □

Proposition 1.2. *If X is a compact space, then there is a metric on $C(X)$.*

Proof. Define $d(f, g) := \sup_{x\in X} |f(x) - g(x)|$. Now, since $f, g \in C(X)$, so does $|f - g|$ because, the absolute value function is continuous. Moreover, we know that continuous image of a compact set is compact. Also, by Heine-Borel theorem a compact subset of $\mathbb{R}$ must be bounded hence, $d : C(X) \times C(X) \to \mathbb{R}$ is well-defined. Now

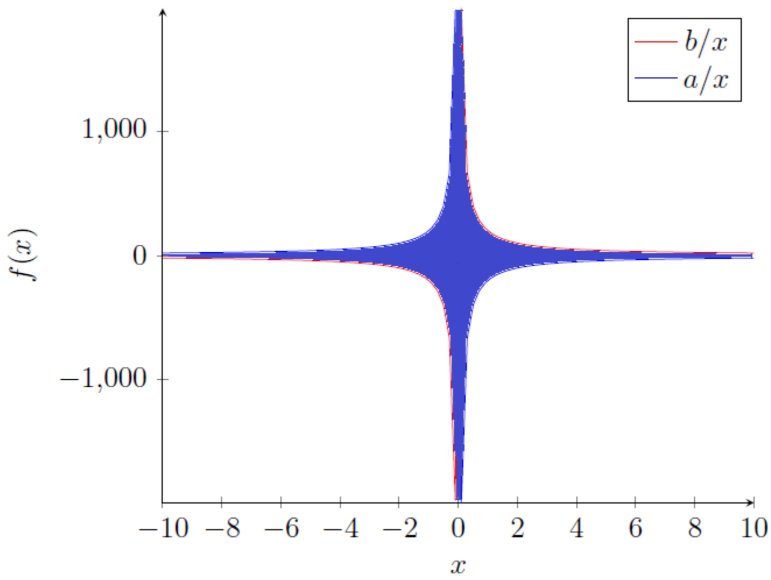


Figure 2: Region 2.

let us verify following properties:
(a) $d(f,g) \geq 0$ for all $f, g \in C(X)$.
(b) $d(f,g) = 0$ if and only if $f = g$.
(c) $d(f,g) = d(g,f)$ for all $f, g \in C(X)$.
(d) $d(f,g) \leq d(f,h) + d(h,g)$ for all $f, g, h \in C(X)$. (Due to the triangle inequality of metric on $\mathbb{R}$ and sup (A+B) $\leq$ (sup (A) + sup (B)). □

Proposition 1.3. *The addition map $+ : C(X) \times C(X) \to C(X)$ and the multiplication map $\times : C(X) \times C(X) \to C(X)$ are continuous (with respect to the metric on $C(X)$).*

Proof. Since, for a metric space continuity is equivalent to sequential continuity. Let $\{(f_n, g_n)\}$ be a sequence in $C(X) \times C(X)$ such that (f_n, g_n) converges to (f, g). Let $\epsilon > 0$ be given, then

$$d((f_n + g_n), (f + g)) \leq d(f_n, f) + d(g_n, g)$$

where, d is the metric on $C(X)$ defined earlier. By hypothesis we can find $N_1, N_2 \in \mathbb{N}$ such that $d(f_n, f) \leq \frac{\epsilon}{2}$ for all $n \geq N_1; (g_n, g) \leq \frac{\epsilon}{2}$ for all $n \geq N_2$ choosing $N = \max(N_1, N_2)$ we get the required convergence.
Similarly, for multiplication, let $\epsilon > 0$ be given. Then,

$$d(f_n.g_n, f.g) \leq d(f_n, f)\|g_n\| + \|f\| d(g_n, g)$$

(Note: $\|f\|$ is the norm of $f \in C(X)$ induced by the metric space, since $C(X)$ is also a vector space which can be verified since all constant functions lie in $C(X)$) and since, every uniformly convergent sequence of bounded functions is uniformly bounded we can say that $\|g_n\| \leq m$ for some $m > 0$. So, there exists $N_1 \in \mathbb{N}$ such

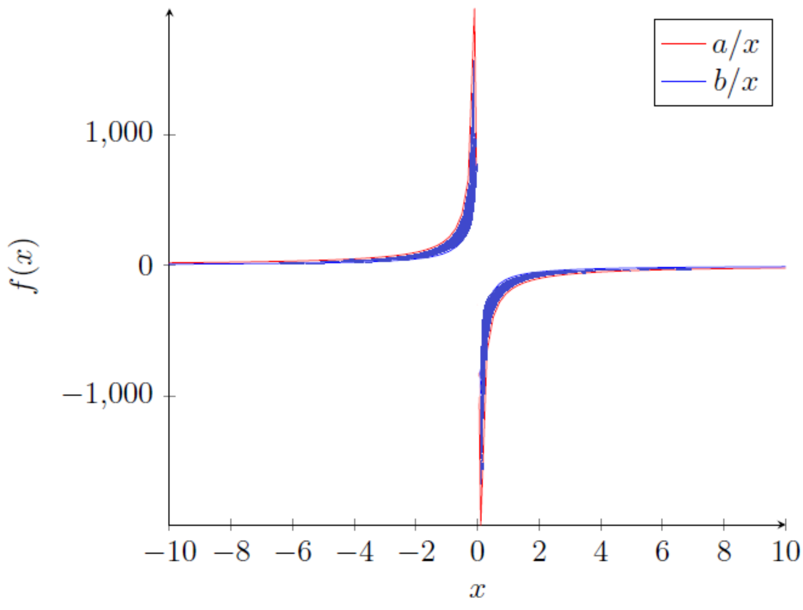


Figure 3: Region 3.

that $d(f_n, f) \leq \dfrac{\epsilon}{2m}$ for all $n \geq N_1$; if $\|f\| = 0$ then choosing $N = N_1$ we are done, otherwise there exists $N_2 \in \mathbb{N}$ such that $d(g_n, g) \leq \dfrac{\epsilon}{2\|f\|}$ for all $n \geq N_2$. Again, choosing $N = \max(N_1, N_2)$ we get the required convergence. □

Proposition 1.4. *If X is a metric space then $C(X)$ separates points.*

Proof. Let $x_0, y_0 \in X$ be given. Consider the following function $f(x) := \dfrac{d(x, x_0)}{d(y_0, x_0)}$. Since x_0 and y_0 are distinct, denominator is never zero, and it is constant (hence, continuous) also, $d(x, x_0)$ is a continuous function of x. So, $f \in C(X)$ and $f(x_0) = 0$ and $f(y_0) = 1$ Hence, it separates the points as well. □

Proposition 1.5. *If $X = \mathbb{R}^n$ then there are polynomials in $C(X)$ that separate points.*

Proof. Let $x_0, y_0 \in \mathbb{R}^n$ be given. Consider the polynomial $f(x) = \dfrac{x - x_0}{\|y_0 - x_0\|}$. This separates x_0 and y_0. □

2 Ring of functions: Banach-Stone theorem

2.1 Compact-open topology

Let X be a topological space. For a compact set $K \subseteq X$ and an open set $V \subseteq \mathbb{R}$ let $U_{K,V}$ consists of all elements $f \in C(X)$ such that $f(K) \subseteq V$. The topology τ_{co} generated by $\{U_{K,V}\}_{K,V}$ as a subbasis is called the compact-open topology, that is,

$$\tau_{co} = \{W \subseteq C(X) \mid W = \bigcup_{\alpha \in I} \bigcap_{i=1}^{n_\alpha} U_{K_{i_\alpha}, V_{i_\alpha}}\}$$

In order to check that a set S $\subseteq \mathbb{P}(X)$ can act as a subbasis of a topology on set X we need to check if $X = \bigcup_{I \in S} I$. Note that this condition gets satisfied if we consider $V = \mathbb{R}$.

Definition 2.1. *We say $\{f_n\} \subseteq C(X)$ converges uniformly over compact sets to f and write $f_n \to f$ if given any compact set $K \subseteq X$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sup_{x \in K} d(f_n(x), f(x)) < \epsilon$ for all $n \geq N$.*

Proposition 2.2. *If $f_n \to f$ and $f \in U_{K,V}$ then there exists $N \in \mathbb{N}$ such that $f_n \in U_{K,V}$ for all $n \geq N$.*

Proof. To show that there exists $N \in \mathbb{N}$ such that $f_n(K) \subseteq V$ for all $n \geq N$. Since $f(K) \subseteq V$, let us consider $V' := f(K)$ which implies V' is compact. Let $d := dist(V^C, V') := \inf\{d(x,y) | x \in V^C, y \in V'\}$ (inf is well defined since the set is non-empty and bounded below by zero). If fact, $d > 0$ since V' is compact, V^C is closed (since it is compliment of an open set) and V' and V^C are disjoint. Choosing ϵ = d, we get a N $\in \mathbb{N}$ from convergence of f_n to f. This will imply $f_n(K) \subseteq V$ for all $n \geq N$. □

Definition 2.3. *A set $A \subseteq C(X)$ is called closed if for any sequence $\{f_n\} \subseteq A$ such that $f_n \to f$ then $f \in A$.*

Proposition 2.4. *These closed sets form a topology, denoted by τ.*

Proof. To show that any collection of closed sets is closed under intersection and any finite collection of closed sets is closed under union. So, let $\{A_\alpha\}_{\alpha \in I}$ be a collection of closed sets. Define $B := \bigcap_{\alpha \in I} A_\alpha$, let $\{f_n\}_{n \in \mathbb{N}} \subseteq B$ such that $f_n \to f$. By the definition of intersection, $f \in B$. Now Let $\{A_i\}_{i=1}^n$ be closed sets. Define $C := \bigcup_{i=1}^n A_i$ and let $\{f_n\}_{n \in \mathbb{N}} \subseteq C$ such that $f_n \to f$. To show that $f \in C$. It is enough to prove for two sets, say A_1 and A_2, as we can extend the result by induction. If $f \in A_1$ or

$f \in A_2$ then clearly $f \in A_1 \cup A_2$. If not, then there exists $\epsilon_{A_1}, \epsilon_{A_2} > 0$ such that for any given compact set $K \subseteq A_1 \sup_{x\in K}\{d(f_n(x), f(x))\} > \epsilon_{A_1}$ for all $n \in \mathbb{N}$ and for any given compact set $K' \subseteq A_2 \sup_{x\in K'}\{d(f_n(x), f(x))\} > \epsilon_{A_2}$ for all $n \in \mathbb{N}$. Now, choose $\epsilon = \min(\epsilon_{A_1}, \epsilon_{A_2})$, then for any compact set $K \subseteq A_1 \cup A_2, \sup_{x\in K}\{d(f_n(x), f(x))\} > \epsilon$ for all $n \in \mathbb{N}$ which contradicts that $f_n \to f$. □

Remark 2.5. *The proof is valid only for finite number of sets since if we consider infinitely many sets then minimum has to be replaced by infimum which might not be positive.*

Proposition 2.6. *The compact open topology is a weaker topology of the usual topology, that is, $\tau_{co} \subseteq \tau$.*

Proof. Let W be a closed set in τ_{co}. Any open set in τ_{co} looks like $\bigcup_{\alpha\in I}\bigcap_{i=1}^{n_\alpha} U_{K_{i_\alpha},V_{i_\alpha}}$.

Hence, any closed of τ_{co} looks like $\bigcap_{\alpha\in I}\bigcup_{i=1}^{n_\alpha}(U_{K_{i_\alpha},V_{i_\alpha}})^C$. We have,

$$U_{K,V} = \{f \in C(X) | f(K) \subseteq V\}$$

Hence, $(U_{K,V})^C = \{f \in C(X) \mid f(x) \in V^C \text{ for some } x \in X\}$. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq A$ such that $f_n \to f$ then, to show that $f \in A$. So we have,

$$f_j \in \bigcap_{\alpha\in I}\bigcup_{i=1}^{n_\alpha}(U_{K_{i_\alpha},V_{i_\alpha}})^C \text{ for all } j \in \mathbb{N}.$$

This in turn implies $f_j \in \bigcup_{i=1}^{n_\alpha}(U_{K_{i_\alpha},V_{i_\alpha}})^C$ for all $j \in \mathbb{N}, \alpha \in I$. Hence, if we fix $j \in \mathbb{N}, \alpha \in I$ then, there exists $i_\alpha \in \mathbb{N}$ such that $f_j \in (U_{K_{i_\alpha},V_{i_\alpha}})^C$. So we can say that

$$\text{there exists } x_j \in K_{i_{\alpha,j}} \subseteq X, V_{i_{\alpha,j}} \subseteq \mathbb{R} \text{ such that } f_j(x_j) \in V^C_{i_{\alpha,j}} \tag{1}$$

To show that $f \in A$, it is enough to show $f \in \bigcup_{i=1}^{n_\alpha}(U_{K_{i_\alpha}}, V_{i_\alpha})^C$, that is to show that there exists $x \in K_{i_\alpha} \subseteq X$ compact, $V_{i_\alpha} \subseteq \mathbb{R}$ open, such that $f(x) \in V^C$. From (1), we have a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \bigcup_{i=1}^{n_\alpha} K_i$, using sequential compactness there a limit point of this sequence, say x and $f_i \to f$ and $x_j \to x$ implies $f_i(x_i) \to f(x)$. So, for given $\alpha \in I$, consider $K := \bigcup_{i=1}^{n_\alpha} K_i, V := \bigcap_{i=1}^{n_\alpha} V_i$ then there exists a point $x \in K$ such that $f(x) \in V^C$. □

If X is compact, by proposition 1.2 we can define metric on $C(X)$, and the space can be denoted as $(C(X), d)$.

Proposition 2.7. *If X is a compact space, then the compact-open topology is equal to the metric topology.*

Proof. Let us first prove $\tau_{co} \subseteq (C(X), d)$. It is enough to show that $U_{K,V}$ is open in $(C(X), d)$. Let $f \in U_{K,V}$, we need to find $r > 0$ such that $B(f, r) \subseteq U_{K,V}$. Define $r := d(f(K), V^C)$. Let $g \in B(f, r)$, to show that $g(K) \subseteq V$. On contrary, let if possible, there exists $x \in X$ such that $g(x) \in V^C$. This implies $|f(x) - g(x)| \geq r$. This gives us a contradiction. Hence inclusion of one side is proved. Now, to prove the reverse inclusion consider an open ball $B(f, r)$. It is enough to prove that there exists $K \subseteq X$ compact and $V \subseteq \mathbb{R}$ open such that $U_{K,V} = B(f, r)$. Let $K = X$, $V = (\|f\| - r, \|f\| + r)$, where $\|f\| = \sup_{x \in X} |f(x)|$. Let $g \in B(f, r)$, we know $\|g(x)\| - \|f(x)\| < |g(x) - f(x)| < r$ which implies, if $|g(x)| > |f(x)|$ then, $|g(x)| < r + |f(x)|$, otherwise $|g(x)| > |f(x)| - r$ for all $x \in X$. Taking supremum we get the inclusion, that is, $B(f, r) \subseteq U_{K,V}$.
For the reverse inclusion let $g \in U_{X,(\|f\|-r,\|f\|+r)}$ and let, if possible $d(g, f) \geq r$, which implies that there exists $x \in X$ such that $|g(x) - f(x)| \geq r$ which contradicts the hypothesis. □

2.2 Stone-Weierstrass Theorem

We will refer [17] for this entire subsection.

Proposition 2.8. *There exists polynomials $\{P_n\}_{n \in \mathbb{N}}$ which converge to e^x.*

Proof. Let $x \in X$. Consider the following sum:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Claim 1: This sum makes sense and we will call the sum as a function denoted by e^x. We will also see that the partial sum is a continuous function and the convergence is uniform hence the limit is also continuous by the uniform convergence theorem.
Proof: We will use root test to prove the convergence.
Claim 2: $(n!)^2 > n^n$ for all $n > 2, n \in \mathbb{N}$.
Proof: For $1 < r < n, r \in \mathbb{N}$, we have $(n - r)(r - 1) > 0$ this implies $r(n - r + 1) > n$. Using this for pairs ((n-2) many of them) summing up to n, we get: $((n-1)!)^2 > n^{n-2}$. This proves the claim 2.
Let $x \in X$, $L := \lim_{k \to \infty} \left| \frac{x^k}{k!} \right|^{1/k} = \lim_{k \to \infty} \left| \frac{x}{(k!)^{1/k}} \right|$ and by claim 2, we have $\frac{1}{\sqrt{k}} > \frac{1}{(k!)^{1/k}} > 0$. Hence, by the comparison test the limit L exists and is 0, that is, $L < 1$. Hence, claim 1 is proved using the root test. □

Lemma 2.9. *The function e^x is not uniformly continuous function over $\mathbb{R}$.*

Proof. Let if possible e^x is uniformly continuous. Then for $\epsilon = 1$, there exists $\delta > 0$ such that $|x - y| < \delta$ which implies $|e^x - e^y| < 1$. Let $a = \dfrac{\delta}{2}$. As $a > 0$ which implies $e^a = 1 + a + \cdots > 1$ which in turn implies $e^a - 1 > 0$ and $\lim_{x \to \infty} e^x = \infty$ which implies $\lim_{x \to \infty} e^x(e^a - 1) = \infty$ which implies there exists $x \in \mathbb{R}$ such that $e^x(e^a - 1) > 1$ but by taking $y = x + a$ we get a contradiction. □

Theorem 2.10 (Weierstrass Approximation Theorem)**.** *Let P be the set of polynomials over the space $X = [0, 1]$. Then the closure of P in $C(X)$ is $C(X)$.*

Proof. Let F: [0,1] $\to \mathbb{R}$ be a continuous function. Then to show that there exists $\{P_n\}_{n \in \mathbb{N}}$ a sequence of polynomials over $[0, 1]$ such that for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(P_n(x), F(x)) < \epsilon$, for all $n \geq N, n \in \mathbb{N}$. Now, without loss of generality, we can assume that $0 < a < b < 1$. Since, $[0, 1]$ and $[a, b]$ are homeomorphic. Note that the polynomials remain polynomials under this homeomorphism. So, let us define

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \dfrac{x \cdot F(a)}{a}, & \text{if } 0 < x \leq a \\ F(x), & \text{if a} < x < \text{b} \\ \dfrac{(1-x)F(b)}{(1-b)}, & \text{if b} \leq x < 1 \\ 0, & \text{if } x \geq 1 \end{cases}$$

Clearly, $f(x) \in C(X)$. Now define,

$$J_n := \int_{-1}^{1} (1 - u^2)^n \cdot du \tag{2}$$

and

$$P_n(x) := \int_{0}^{1} f(t)(1 - (t - x)^2)^n \cdot dt$$

Maximum power of x in $P_n(x)$ is 2n, and all the functions inside the integrand are continuous so, $P_n(x)$ is indeed a polynomial. Now, replace the limits of $P_n(x)$ by $(-1 + x)$ and $(1 + x)$ as $f(t)$ is zero outside (0,1).

Which implies, $P_n(x) = \displaystyle\int_{-1+x}^{1+x} f(t)(1 - (t - x)^2)^n \cdot dt.$

Substituting $t - x = u$ we get $P_n(x) = \displaystyle\int_{-1}^{1} f(x + u).(1 - u^2)^n \cdot du.$

From equation (2), we get $f(x) := \dfrac{1}{J_n} \displaystyle\int_{-1}^{1} f(x)(1 - u^2)^n \cdot du$

Which implies, $P_n(x) - f(x) = \dfrac{1}{J_n} \displaystyle\int_{-1}^{1} [f(x + u) - f(x)](1 - u^2)^n \cdot du.$

As $f(x)$ is zero outside the compact set [0,1] and f is continuous implies f is uniformly continuous, which implies that there exists $\delta > 0$ such that

$$|f(x + u) - f(x)| < \epsilon, \text{ for all } |u| \leq \delta \tag{3}$$

and let $M := \sup_{x \in [0,1]} |f(x)|$ which implies, $|f(x+u) - f(x)| < 2M,$ for all $|u| \geq \delta$.
This implies,

$$|f(x+u) - f(x)| < 2M \cdot \frac{u^2}{\delta^2} \tag{4}$$

Combining (3) and (4), we get:

$$|P_n(x) - f(x)| \leq \frac{1}{J_n} \int_{-1}^{1} \frac{\epsilon}{2}(1-u^2)^n \cdot du + \frac{1}{J_n} \int_{-1}^{1} \frac{2Mu^2}{\delta^2}(1-u^2)^n \cdot du$$

$$= \frac{\epsilon}{2} + \frac{2M}{\delta^2 J_n} \int_{-1}^{1} u^2(1-u^2)^n \cdot du$$

Consider $J_n' := \int_{-1}^{1} u^2(1-u^2)^n \cdot du = \frac{J_{n+1}}{2(n+1)}$ (using by parts).
Since, $1-u^2 < 1$ which implies, $J_{n+1} < J_n$, which implies, $2 \cdot J_n' \cdot (n+1) < J_n$, which implies, $\frac{J_n'}{J_n} < \frac{1}{2(n+1)}$, for all $n \in \mathbb{N}$. This implies that there exists $N \in \mathbb{N}$ such that $\frac{J_n'}{J_n} < \frac{\delta^2}{2M} \frac{\epsilon}{2}$ for all $n \geq N, n \in \mathbb{N}$. Hence, $|P_n(x) - f(x)| < \epsilon$ for all $n \geq N, n \in \mathbb{N}$. □

Corollary 2.11. *For every interval $[-a, a]$ there is a sequence of real polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $P_n(0) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} P_n(x) = |x|$ uniformly on $[-a, a]$.*

Proof. As $|x|$ is a continuous function for $x \in [-a, a]$, we have a sequence of polynomials say, $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ by theorem 2.10. Define $P_n(x) := \tilde{P}_n(x) - \tilde{P}_n(0)$ for all $n \in \mathbb{N}$. Since $\tilde{P}_n(0)$ is just a constant, $P_n(x)$ is also a polynomial which gives the desired sequence. □

Definition 2.12 (Algebra). *A family A of real (or complex) valued functions defined on a set E is said to be an algebra if*
(i) $f + g \in A$ for all $f, g \in A$.
(ii) $f \cdot g \in A$ for all $f, g, \in A$.
(iii) $c \cdot f \in A$ for all $f \in A, c \in \mathbb{R}$(or $\mathbb{C}$).

Definition 2.13 (Uniform closure of an algebra). *B is the uniform closure of an algebra A if B is the set of all functions which are limits of uniformly convergent sequences of elements of A and it is denoted by $\overline{A}$, that is, $\overline{A} := \{f \in C(X) \mid f_n \to f$ for a sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq A\}$.*

Proposition 2.14. *Let B be the uniform closure of an algebra A of bounded functions. Then B is a uniformly closed algebra.*

Proof. The only thing we have to show is that B is an algebra. So, let $f, g \in B, c \in \mathbb{R}$. Then, there exists sequences $\{f_n\}_{n \in \mathbb{N}}$, $\{g_n\}_{n \in \mathbb{N}} \subseteq A$ such that $f_n \to f$ uniformly and $g_n \to g$ uniformly. It is clear that using boundedness $f_n + g_n \to f + g$, $f_n \cdot g_n \to f \cdot g$ and $c \cdot f_n \to c \cdot f$ since $\{f_n + g_n\}$, $\{f_n \cdot g_n\}$, $\{c \cdot f_n\}$ are sequences in A (as A is an algebra) and B is uniformly closed, hence $f + g, f \cdot g, c \cdot f \in B$. □

Definition 2.15. *Let A be a family of functions on a set E.*
(i) Then A is said to separate points on E if, for each $x_1, x_2 \in E(x_1 \neq x_2)$, there exists f in A such that $f(x_1) \neq f(x_2)$.
(ii) We say that A vanishes at no point of E, if for each $x \in E$, there exists $g \in A$ such that $g(x) \neq 0$.

Proposition 2.16. *Let A be an algebra on a set E. A separates points on E. A vanishes at no point of E. Suppose x_1, x_2 are distinct points and c_1, c_2 are constants, then there exists $f \in A$ such that $f(x_1) = c_1, f(x_2) = c_2$.*

Proof. Assumptions imply that there exists $g, h, k \in$ A such that $g(x_1) \neq g(x_2), h(x_1) \neq 0, k(x_2 \neq 0)$. Now define $u := gk - g(x_1)k, v := gh - g(x_2)h$. Now, consider $f := \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$ which implies $f \in$ A and $f(x_1) = c_1, f(x_2) = c_2$. □

Theorem 2.17 (Stone-Weierstrass theorem)**.** *Let A be an algebra of real continuous functions on a compact set K. If A separates points on K and if A vanishes at no point of K, then the uniform closure B of A consists of all real continuous functions on K.*

Proof. This proof consists of 4 steps.
Claim 1: $f \in B$ implies $|f| \in B$.
Proof:

$$a := \sup_{x \in K} |f(x)| \tag{5}$$

Note that this is well defined since K is compact and $|f|$ is continuous since f is continuous. Let $\epsilon > 0$ be fixed. Then by corollary 2.11, there exists $C_1, C_2, \cdots, C_n \in \mathbb{R}$ such that

$$\left| \sum_{i=1}^{n} C_i y^i - |y| \right| < \epsilon, \text{ for all } y \in [-a, a] \tag{6}$$

Since B is an algebra, $g := \sum_{i=1}^{n} C_i f^i \in B$. By (5) and (6), we have $\|g(x) - f(x)\| < \epsilon$ for all $x \in K$. This implies $|f|$ is a limit point of A, hence $f \in B$. Since B is the uniform closure of A. This proves claim 1.
Claim 2: $f, g \in B$ implies $\max(f, g), \min(f, g) \in B$.
Proof: Note that $\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$ and $\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$. By claim 1, we can say that $|f - g| \in B$ and B being an algebra $\max(f, g)$ and $\min(f, g) \in B$. Note that the same result can be extended for finitely many functions by using method of induction.
Claim 3: Let f be a continuous function on K, $x \in X, \epsilon > 0$ be fixed, then there exists $g_x \in B$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$ for all $t \in K$.
Proof: Note that $A \subseteq B$. Hence, due to proposition 2.16 for each $y \in K$, there exists $h_y \in B$ such that $h_y = f(x), h_y = f(y)$. Using continuity of h_y, there exists an open set J_y containing y such that $h_y > f(t) - \epsilon$ for all $t \in J_y$. Now, $\{J_y\}_{y \in K}$ form an open cover of K and as K is compact, we can write $K \subseteq \bigcup_{i=1}^{n} J_{y_i}$ for some $n \in \mathbb{N}$. Define:

$g_x := \max(h_{y_1}, h_{y_2}, \cdots, h_{y_n})$. Using claim 2, $g_x \in B$, also, $g_x(t) > f(t) - \epsilon$ for all $t \in K$.
Claim 4: Given a real valued function f, continuous on K, $\epsilon > 0$ be fixed. Then there exists $h \in B$ such that $|h(x) - f(x)| < \epsilon$ for all $x \in K$.
Proof: Consider g_x constructed in claim 3. Using continuity of g_x, there exists an open set V_x containing x such that $g_x(t) < f(t) + \epsilon$ for all $t \in V_x$.

Again, using compactness of K we can get $x_1, x_2, \cdots, x_m$ such that $K \subseteq \bigcup_{i=1}^{m} V_{x_i}$.
Define $h := \min(g_{x_1}, g_{x_2}, \cdots, g_{x_m})$. Using claim 2, $h \in B$ and

$$h(t) < f(t) + \epsilon \text{ for all } t \in K \tag{7}$$

Now, using claim 3 and (7) we have $h \in C(K)$ such that $|f(x) - h(x)| < \epsilon$ for all $x \in K$. □

Definition 2.18 (Self-adjoint)**.** *Let A be a set of complex valued functions then it is called self-adjoint if each $f \in A$ implies $\overline{f} \in A$ for all $f \in A$ where, $\overline{f}(x) := \overline{f(x)}$ for all x in the domain of f.*

Corollary 2.19. *Let A be a self-adjoint algebra of complex valued continuous funtions on K such that A separates points on K, and A vanishes at no point of K. Then $\overline{A}$ consists of all complex valued continuous functions on K.*

Proof. Let $A_{\mathbb{R}}$ be the set of all real valued functions on K which belong to A. Let $f \in A$, then $f = u + iv$ such that $u, v \in A_{\mathbb{R}}$. A is self-adjoint hence $f + \overline{f} = 2u \in A_{\mathbb{R}}$ which implies, $u \in A_{\mathbb{R}}$ similarly, $v \in A_{\mathbb{R}}$. Also, $x_1 \neq x_2$ so, there exists $f \in A$ such that $f(x_1) = 1, f(x_2) = 0$. Hence, $u(x_2) = 0, u(x_1) = 1$ which implies $A_{\mathbb{R}}$ separates points on K. If $x \in K$, then there exists $g \in A$ such that $g(x) \neq 0$, and $\lambda \in \mathbb{C}$ such that $\lambda \cdot g(x) \in \mathbb{R}$ and is positive. Define $f := \lambda \cdot g$, and $f = u + iv$, then $u(x) > 0$ which implies, $A_{\mathbb{R}}$ vanishes at no point of K. Now, using theorem 2.17 separately for u, v we get that $u \in B, v \in B$ which implies $f \in B$. □

2.3 Maximal ideals in $C(X)$

Theorem 2.20. *[4] Let X be a compact topological space. Let $x \in X$ be a point. Then, $\mathfrak{m}_x := \{f \in C(X) | f(x) = 0\}$ is a maximal ideal in $C(X)$. Moreover, any maximal ideal is $\mathfrak{m}_x$ for some $x \in X$.*

Proof. By the definition of addition and multiplication in $C(X)$ given in proposition 1.1 it is evident that $\mathfrak{m}_x$ is indeed an ideal. Since it is closed under addition and $f \cdot g \in \mathfrak{m}_x$ for all $f \in C(X)$, $g \in \mathfrak{m}_x$. Now, let J be an ideal in $C(X)$ such that $\mathfrak{m}_x \subsetneq J$ implies such that $f(x) \neq 0$, for some $f \in J$ define $g \in C(X)$ as follows: $g(y) := \dfrac{1}{f(x)}$ for all $y \in X$ and $h := (f \cdot g - 1)$ then, $h \in \mathfrak{m}_x$, and as $\mathfrak{m}_x \subsetneq J, h \in J$. This implies, $f \cdot g - h = 1 \in J$. Hence, $J = C(X)$. Note that 1 is the constant 1

function in $C(X)$. For the second part, let $\mathfrak{m}$ be a maximal ideal in $C(X)$. Then for $K \subseteq C(X)$, define $V(K) := \{x \in X | f(x) = 0, \text{ for all } f \in K\}$, and for $A \subseteq X$, define $I(A) := \{f \in C(X) | f(x) = 0 \text{ for all } x \in A\}$. Now we claim that $V(\mathfrak{m}) \neq \varnothing$.
Let if possible $V(\mathfrak{m}) = \varnothing$. This implies for each $x \in X$ there exists $f_x \in C(X)$ such that $f_x(x) \neq 0$. Now, define $U_x := \{y \in X | f_x(y) \neq 0\}$ which implies $x \in U_x$. Hence, $\{U_x\}_{x \in X}$ forms a cover of X and due to compactness of X there exists $\{U_{x_i}\}_{i=1}^{n}$ a finite subcover. Then the function $(f_{x_1}^2 + f_{x_2}^2 + \cdots + f_{x_n}^2)(x) \neq 0$ for all $x \in X$. Hence, it is a unit, and so, $\mathfrak{m} = C(X)$. This is a contradiction. Hence, $V(\mathfrak{m}) \neq \varnothing$. Let, $x \in V(\mathfrak{m})$, now $K \subseteq I(V(K))$, for all $K \subseteq C(X)$ and $K_1 \subseteq K_2$ which implies $I(K_2) \subseteq I(K_1)$ which in turn implies $\mathfrak{m} \subseteq I(V(\mathfrak{m})) \subseteq I(x) = \mathfrak{m}_x$ and $\mathfrak{m}_x \neq C(X)$ as constant functions do not belong to $\mathfrak{m}_x$. Hence, as $\mathfrak{m}$ is maximal, $\mathfrak{m} = \mathfrak{m}_x$. □

From the theorem 2.20, we get to know that, if a topological space X is compact and Hausdorff, then there is a one-to-one correspondence between the points of the space X and maximal ideals of $C(X)$. So, it is natural to seek for examples where the correspondence does not hold when we relax the condition of compactness or Hausdorffness.

Example 2.21. *Let X be $(0,1)$. Clearly, $(0,1)$ is not compact, so we claim that there exists a maximal ideal which is not of the form $\mathfrak{m}_x$ for any $x \in (0,1)$. Consider, $I := \{f \in C((0,1)) \mid f\left(\frac{1}{n}\right) = 0$ for all but finitely many $n \in \mathbb{N}\}$. Observe that I is an ideal since it is closed under addition and $r \cdot a \in I$ for all $r \in C((0,1)), a \in I$. Also, I is a proper ideal simply because the constant one function does not belong to I. Consider the following function, for $m \in \mathbb{N}$.*

$$g_m(x) = \begin{cases} sin\left(\frac{\pi}{x}\right), & \text{if } x \leq \frac{1}{m} \\ \\ x - \frac{1}{m}, & \text{if } x > \frac{1}{m} \end{cases}$$

Observe that this function is continuous everywhere, in particular at $x = \frac{1}{m}$. Also, $g_m(x) = 0$ if and only if $x = \frac{1}{n} \leq \frac{1}{m}$ that is $m \leq n$. So, $g_m(x) \in I$. Now, give any maximal ideal of the form $\mathfrak{m}_a$, if we choose m big enough such that $\frac{1}{a} \leq m$ then as we saw above $g_m(a) \neq 0$, and thus $g_m(x) \notin \mathfrak{m}_a$. Finally, we use Zorn's lemma to remember that $I \subseteq M$, for some maximal ideal $M \subseteq C((0,1))$ [1] and we have seen that $M \neq \mathfrak{m}_x$ for any $x \in (0,1)$.

Example 2.22. *When $X = \mathbb{N}$, with the usual subspace topology inherited from $\mathbb{R}$. Again $\mathbb{N}$ is a non compact space. Note that an element in $C(\mathbb{N})$ is nothing but a sequence of real numbers since the topology on $\mathbb{N}$ is the discrete topology. Consider, $I := \{(a_1, a_2, \cdots) \mid a_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and a_i's are 0 for all but finitely many i's $\}$. Again we will use the same procedure as the previous example and say that I is an ideal, moreover it is a proper ideal. Also, consider the element $b_j = (a_1, a_2, \cdots)$ such that $a_i = 0$ for all $i \neq j$, and $a_j = 1$. Then $b_j \in I$ and $b_j \notin m_j$ and again using Zorn's lemma, we get a maximal ideal M containing I with the property that $M \neq \mathfrak{m}_x$ for any $x \in \mathbb{N}$.*

Example 2.23. *Let X be a set with more than one point. A natural non-Hausdorff topology one can think about is the indiscrete topology on X. In that case, the only elements of $C(X)$ are the constant functions and we know that we can not have any unit in the ideal in order to have it as a proper ideal. So, the zero ideal is the only proper ideal in $C(X)$, which, therefore is maximal ideal. So, the one-to-one correspondence between points of X and maximal ideals of $C(X)$ fails by the cardinality argument, since there is only one maximal ideal in $C(X)$ and there are more than one points in X.*

For (X, τ) compact Hausdorff, we have a map ϕ: $X \to maxspec(C(X))$, due to theorem 2.20, where $x \mapsto \mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}$. We claim that ϕ is a continuous map where $maxpsec(C(X))$ is equipped with the Zariski topology (defined in the appendix A).
Consider a closed set $V(E) \subseteq maxspec(C(X))$. Using theorem 2.20 we can write $V(E) = \{\mathfrak{m}_x \mid E \subseteq \mathfrak{m}_x, x \in X\}$. We claim that $\phi^{-1}(V(E)) = \{x \in X | f(x) = 0,$ for all $f \in E\}$. First, let $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in E$. To prove that $x_0 \in \phi^{-1}(V(E))$, that is, to prove that $\phi(x_0) \in V(E)$ that is $\mathfrak{m}_{x_0} \in V(E)$, that is, $E \subseteq \mathfrak{m}_{x_0}$. So, let $f \in E \subseteq C(X)$, then we have $f(x_0) = 0$ which implies $f \in \mathfrak{m}_{x_0}$. This takes care of one inclusion. Now, for the reverse inclusion, let $y \in \phi^{-1}(V(E))$, which implies $\phi(y) \in V(E)$ which in turn implies $\mathfrak{m}_y \in V(E)$ hence, $E \subseteq \mathfrak{m}_y$. So, letting $f \in E$ we get $f(y) = 0$ since $E \subseteq \mathfrak{m}_y$. This completes the proof of the claim. Now, it is easy to see that $\{x \in X \mid f(x) = 0$ for all $f \in E\}$ is closed set in τ. For some $f \in E$ the set $\{x \in X \mid f(x) = 0\}$ is a closed set, since f is continuous and $\{0\}$ is a closed set. Now, $\{x \in X \mid f(x) = 0$ for all $f \in E\}$ is just the intersection of closed sets which is also a closed set. Hence, ϕ is a continuous map. Now we claim that ϕ is more than a continuous map, it is a homeomorphism.

Lemma 2.24. *Let X be a compact space and Y be a Hausdorff space and $f : X \to Y$ be a continuous bijection. Then f is a homeomorphism.*

Proof. It remains to prove that the map f is an open map. But since f is a bijection it is enough to show that f is a closed map, meaning f sends closed sets to closed sets. So, let $C \subseteq X$ be closed. Since X is compact C is also compact. Hence the continuous image $f(C)$, of C is also compact. Now, since Y is Hausdorff, the compact subset $f(C)$ is closed. □

Proposition 2.25. *Let X be a compact Hausdorff space, then ϕ: $X \to maxspec(C(X))$ where $x \mapsto \mathfrak{m}_x$, is a homeomorphism.*

Proof. Let us first prove that ϕ is injective. Let $x, y \in X$ be distinct points. Since X is compact Hausdorff, it is normal. So, using Urysohn's lemma there exists a continuous function f_x such that $f_x(x) = 0, f_x(y) = 1$ which implies $f_x \in \mathfrak{m}_x$ but $f_x \notin \mathfrak{m}_y$ which in turn implies $\phi_x = \mathfrak{m}_x \neq \mathfrak{m}_y = \phi(y)$. This prove the injectivity of ϕ. Surjectivity of ϕ comes directly from theorem 2.20. So, to prove this proposition using the previous lemma it remains to prove that $maxspec(C(X))$ is Hausdorff. Since we have proved that ϕ is a bijection, two distinct points of $maxspec(C(X))$ can be considered as $\mathfrak{m}_x$ and $\mathfrak{m}_y$, where x, y are distinct points in X. It is enough to show that there exists $f, g \in C(X)$ such that $\mathfrak{m}_x \notin V(\langle f \rangle), \mathfrak{m}_y \notin V(\langle g \rangle)$, and $V(\langle f \rangle) \cup V(\langle g \rangle) = maxspec(C(X))$. Now, since X is Hausdorff, there exists disjoint open sets U_x, U_y containing x, y respectively. Since compact Hausdorff spaces are normal and normal spaces are Tychonoff spaces, there exist continuous functions f, g such that $f(x) = 1, f(U_x^C) = \{0\}$, similarly, $g(y) = 1, g(U_y^C) = \{0\}$. Now see that $\langle f \rangle \not\subset \mathfrak{m}_x$ since $\mathfrak{m}_x$ is the set of all continuous functions vanishing at x, but f does not vanish at x, hence $\mathfrak{m}_x \notin V(\langle f \rangle)$, similarly $\mathfrak{m}_y \notin V(\langle g \rangle)$. Now to prove that $V(\langle f \rangle) \cup V(\langle g \rangle) = maxspec(C(X))$. One inclusion is obvious. For the other one, let P be a maximal ideal in $C(X)$. Observe that $U_x^C \cup U_y^C = X$ which implies $f \cdot g \equiv 0$ on X and we know $\langle f \cdot g \rangle = \langle f \rangle \langle g \rangle$ and every ideal contains $\langle 0 \rangle$. So, $\langle f \rangle \langle g \rangle = \langle 0 \rangle \subseteq P$. Now, since maximal ideal is a prime ideal, either $\langle f \rangle \subseteq P$ or $\langle g \rangle \subseteq P$ or both, which implies $P \in V(\langle f \rangle)$ or $P \in V(\langle g \rangle)$ or both. This makes $maxspec(C(X))$ Hausdorff and ϕ a homeorphism. □

Since we are discussing relationships between two collections of objects, namely, compact Hausdorff topological spaces and rings of continuous functions over compact Hausdorff spaces, one can naturally ask if the equivalence of structures preserved when we go from one side to other. This leads us to a version of the statement of Banach-Stone theorem.

Theorem 2.26 (Banach-Stone theorem)**.** *[5, 21] Let X and Y be compact, Hausdorff spaces, then X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic as rings.*

Proof. ($\Rightarrow$) This side is trivial, since given a homeomorphism $f : X \to Y$ one can define a map $\phi : C(Y) \to C(X)$ such that $(g : Y \to \mathbb{R}) \mapsto (g \circ f : X \to \mathbb{R})$. It is easy to check that this map is indeed a ring homorophism, and one can define a map $\psi : C(X) \to C(Y)$ such that $(g : X \to \mathbb{R}) \mapsto (g \circ f^{-1} : Y \to \mathbb{R})$ which implies $\phi \circ \psi = Id, \psi \circ \phi = Id$. Hence, $C(X)$ and $C(Y)$ are indeed isomorphic.
($\Leftarrow$) Let $C(X)$ and $C(Y)$ be isomorphic, which implies $maxspec(C(X))$ and $maxspec(C(Y))$ are homeomorphic, hence invoking proposition 2.25 we get X and Y to be homoeomorphic. This completes the proof. □

Lemma 2.27. *Let X, Y be topological spaces and $f : X \to Y$ be a homeomorphism, then open sets of X and Y are in one-to-one correspondence with each other.*

Proof. We claim that any open set $U \subseteq X$ can be written as $U = f^{-1}(V)$ for a unique open set $V \subseteq Y$. Let $U \subseteq X$ be a fixed open set. Considering $V = f(U)$, we get $U = f^{-1}(V)$. So, the existence part is clear. Now, let there exist two such open sets V_1, V_2. Since f is an injection, $f \circ f^{-1}$ is an identity map. Hence, applying f to $f^{-1}(V_1) = U = f^{-1}(V_2)$, we get $V_1 = V_2$. This completes the proof. □

Taking motivation from proposition 2.25 we can put a weaker topology on a space X, which we will call the Zariski topology on X and denote it by $\tau_{Zariski}$. A set $C \subseteq X$ is said to be closed if there exists an ideal $J \subseteq C(X)$ such that $C = Z(J) := \{x \in X \mid f(x) = 0 \text{ for all } f \in J\}$. Let us first check that this indeed forms a topology.

For $C = \varnothing$, we have $J = C(X)$, since $C(X)$ contains non-zero constant functions. For $C = X$, we have $J = \langle 0 \rangle$. Now let $\{Z(J_i)\}_{i \in I}$be an arbitrary collection of closed sets. Consider K to be the smallest ideal containing $\bigcup_{i \in I} J_i$, that is, we claim that $\bigcap_{i \in I} Z(J_i) = Z(K)$. Let $x \in \bigcap_{i \in I} Z(J_i)$, which implies $x \in Z(J_i)$ for all $i \in I$ hence $f(x) = 0$ for all $f \in J_i, i \in I$ which implies $f(x) = 0$ for all $f \in \bigcup_{i \in I} J_i$. Now it can be easily seen by method of contradiction that $f(x) = 0$ for all $f \in K$, that is $x \in Z(K)$. Now, for the reverse inclusion, let $x \in Z(K)$, which implies $f(x) = 0$ for all $f \in K$ and since $J_i \subseteq K$ for all $i \in I$, we have $f(x) = 0$ for all $f \in J_i, i \in I$ which implies $x \in \bigcap_{i \in I} V(J_i)$. Now, for the closure under finite union, let $K = \bigcap_{i=1}^{n} J_i$. Let $x \in \bigcup_{i=1}^{n} Z(J_i)$, which implies $x \in Z(J_{i_0})$ for some $i_0 \in \{1, 2, \cdots, n\}$. So, $f(x) = 0$ for all $f \in J_{i_0}$ and as $K \subseteq J_{i_0}$, we have $f(x) = 0$ for all $f \in K$, hence $x \in Z(K)$. Now, let $x \in Z(K)$. Let if possible, there does not exist any i such that $x \in Z(J_i)$. So, for each $i \in \{1, 2, \cdots, n\}$ there exists $f_i \in J_i$ such that $f_i(x) \neq 0$. Then consider $f := \prod_{i=1}^{n} f_i \in K$ and the construction implies $f(x) \neq 0$ but $f(x) = 0$ since $f \in K$ and $x \in Z(K)$. This gives us a contradiction. Hence, it is indeed a topology on X.
Note that, if (X, τ) any topological space, then $\tau_{Zariski} \subseteq \tau$ simply because any closed set in $\tau_{Zariski}$ is intersection of $\{x \in X \mid f(x) = 0\}$ for some $f \in C(X)$ which are closed sets in τ. But when the topological space X is compact Hausdorff, from proposition 2.25, lemma 2.27 and the definition of Zariski topology on X we get $\tau = \tau_{Zariski}$.

3 Line bundles and double covers

In this chapter, we will be studying the following objects: double covers of certain topological spaces, index two subgroups of the fundamental groups of some topological spaces, and line bundles. We will establish one-to-one correspondences among these structures when the underlined topological space is same. We will see that these correspondences will agree over the isomorphism classes of each object. The constraints on the space for respective correspondences are discussed in detail in the chapter, since the correspondences are not valid for any general topological space. We will refer [9, 20] for this chapter.

3.1 Definitions and Preliminaries

Proposition 3.1. *A space X is simply connected if and only if there is a unique based homotopy class of path connecting any two points in X.*

Proof. We know that the definition of simply connectedness inherently assumes the path-connectedness.
(⇒) Let $\pi_1(X) = 0$. If f and g are two paths from x_0 to x_1, then f is homotopic to $f\overline{g}g$ and similarly, $f\overline{g}g$ is homotopic to g, since, the loops $\overline{g}g$ and $f\overline{g}$ are each homotopic to constant loops.
(⇐) If there is only one homotopy class of path connecting a basepoint x_0 to itself, then all loops based at x_0 are homotopic to the constant loop. Hence, $\pi_1(X, x_0) = 0$. □

Proposition 3.2. *A subgroup H of index two of a group G is normal.*

Proof. First, let us recall that the cosets partition the group. So if there are only two cosets, one of which is the subgroup itself, then the second coset must be the remaining elements other than H. So the cosets of H in G are H and H^C. Let $g \in G$. Case 1: $g \in H$. Then $gH = H = Hg$. Case 2: $g \notin H$. Then $gH \neq H$ and so $gH = H^C$. Likewise, $Hg \neq H$ so, $Hg = H^C$. Therefore, $gH = Hg$. Hence, H is normal in G. □

Definition 3.3 (Locally path-connected)**.** *A topological space X is said to be locally path-connected, if for any given point $x \in X$ and an open set U there exists a open set V such that $x \in V \subseteq U$ and V is path-connected.*

Definition 3.4 (Semi-locally simply connected)**.** *A topological space X is said to be semi-locally simply connected, if for any given point $x \in X$ there exists an open set U containing x such that any loop in U is nullhomotopic in X.*

Definition 3.5 (Double cover)**.** *Let X be a topological space. A covering space of X is a topological space $\tilde{X}$ together with a continuous surjective map $p : \tilde{X} \to X$, such that for each $x \in X$, there exists an open neighborhood U containing x, such that $p^{-1}(U)$ is a disjoint union of two open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p.*

Definition 3.6 (Isomorphism of covering spaces). *Two covering spaces ($\tilde{X}_1, p_1$), ($\tilde{X}_2, p_2$) over same topological space X are said to be isomorphic, if there exists a homeomorphism f: $\tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ f$.*

Definition 3.7 (Trivial double cover). *Let X be a topological space, then $\tilde{X} := X \sqcup X = X \times \{0\} \cup X \times \{1\}$ and $p((x, 0)) := x, p((x, 1)) := x$ for all $x \in X$ forms a double cover. This is called the trivial double cover.*

Proposition 3.8 (Lifting criterion). *Suppose we are given a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. ($\Rightarrow$) This is immediate from $f = p \circ \tilde{f}$, which implies $f_* = p_* \circ \tilde{f}_*$.
($\Leftarrow$) Let $y \in Y$, γ be a path in Y from y_0 to y. Consider the path $f\gamma$ in X starting at x_0 and its unique lift $\widetilde{f\gamma}$ starting at $\tilde{x_0}$. Define $\tilde{f}(y) := \widetilde{f\gamma}(1)$ for all y in Y. In order to show that this $\tilde{f}$ is well-defined (that is, it is independent of choice of γ) consider γ' another path from y_0 to y. Then $(f\gamma')(\overline{f\gamma})$ is a loop (say h_0) at x_0 with $[h_0] \in f_*(\pi_1(Y, y_0))$ and since, $f_*(\pi_1(Y, y_0)) \subseteq P_*(\pi_1(\tilde{X}, \tilde{x_0}))$ there is a homotopy h_t of h_0 to a loop h_1 such that $\tilde{h}_1$ is a loop at $\tilde{x}_0$, so is $\tilde{h}_0$. By uniqueness of lifted paths, first half of $\tilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\widetilde{\overline{f\gamma}}$ with common mid-point $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. Hence, $\tilde{f}$ is well-defined. Now, to show that $\tilde{f}$ is continuous, let $U \subseteq X$ be an open neighbourhood of $f(y)$ having a lift $\tilde{U} \subseteq \tilde{X}$ containing $\tilde{f}(y)$ such that $p : \tilde{U} \to U$ is homeomorphism. Now, choose a path-connected neighbourhood V of y with $f(V) \subseteq U$. (Note that such a neighbourhood exists due to continuity of f and local path-connectedness of Y). For paths from y_0 to $y'(\in V)$, consider the path which is γ in the beginning from y_0 to y; followed by a path η in V from y to y'. Then, $(f\gamma)(f\eta)$ in X will have lifts $(\widetilde{f\gamma})(\widetilde{f\eta})$ where, $\widetilde{f\eta} = p^{-1}(f\eta)$. Thus $\tilde{f}(V) \subseteq \tilde{U}$ and $\tilde{f}|_V = p^{-1} \circ f$. Hence, $\tilde{f}$ is continuous at y since, p^{-1}, f are continuous on $f(V)$ and V respectively. ☐

Proposition 3.9 (Unique lifting property). *Given a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$ with two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ that agree on one point of Y, then if Y is connected, these two lifts must agree on all points in Y.*

Proof. Let $y \in Y$ be a point. Let U be an open neighbourhood of $f(y)$ in X for which $p^{-1}(U)$ is a disjoint union of open sets $\tilde{U}_\alpha$ each mapped homeomorphically to U by p. Let $\tilde{U}_1$ and $\tilde{U}_2$ be the sets containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ respectively. By continuity of $\tilde{f}_1$ and $\tilde{f}_2$ there is a neighbourhood N of y mapped into $\tilde{U}_1$ by $\tilde{f}_1$ (that is, $\tilde{f}_1(N) \subseteq \tilde{U}_1$) and into $\tilde{U}_2$ by $\tilde{f}_2$. Since, $\tilde{U}_1, \tilde{U}_2 \in \{U_\alpha\}$, $\tilde{U}_1$ and $\tilde{U}_2$ can either be disjoint or equal. So, if $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then they are disjoint, hence $\tilde{f}_1 \neq \tilde{f}_2$ throughout N. On the other hand, if $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_1 = \tilde{U}_2$. Now, since $\tilde{f}_1, \tilde{f}_2$ are lifts of f, and $p \circ \tilde{f}_1 = f = p \circ \tilde{f}_2$ and since, p is injective on $\tilde{U}_1 (= \tilde{U}_2)$, hence $\tilde{f}_1 = \tilde{f}_2$ on N. Thus, the set N is both open and closed (and non-empty) hence, using connectedness of Y, we are done. ☐

Proposition 3.10. *Let $p : X \to Y$ be a covering map. Then the induced map $p_* : \pi_1(X, x_0) \to \pi_1(Y, p(x_0))$ is an injective map.*

Proof. First consider the statement of homotopy lifting property. Let $p : X \to Y$ be a covering map, and let $f_t : Z \to Y$ be a homotopy, with $\tilde{f}_0 : Z \to X$ a lift of f_0. Then there is a unique homotopy $\tilde{f}_t : Z \to X$ of $\tilde{f}_0$ lifting f_t. Now, let $Z = I$, so that $\tilde{f}_0$ is a path in X. Suppose that $p \circ \tilde{f}_0 = f_0$ is trivial in $\pi_1(Y, p(x_0))$, so that we have a homotopy $f_t : I \to Y$ taking f_0 to the constant path f_1. By the homotopy lifting property, this gives us a homotopy $\tilde{f}_t$ which takes $\tilde{f}_0$ to a lift of the constant path. By uniqueness, a lift of the constant path in Y is the constant path in X, so that in fact $\tilde{f}_0$ is trivial in $\pi_1(X, x_0)$. □

Definition 3.11 (Vector bundle). *A vector bundle over a topological space X is a pair (E, π) satisfying the following conditions:*
(i) E is a topological space, which is also called total space.
(ii) $\pi : E \to X$ is a continuous surjective map, which is also called the projection of vector bundle.
(iii) There is a fixed $r \in \mathbb{N}$ (rank of E) such that for each $p \in X$, $\pi^{-1}(p)$ is an r-dimensional vector space over $\mathbb{R}$. Moreover, $\pi^{-1}(p)$ is called fiber over p denoted by E_p.
(iv) Condition of local triviality: For each point $p \in X$, there is a neighbourhood U of p which is also called as trivializing neighbourhood and a homeomorphism $\phi : U \times \mathbb{R}^r \to \pi^{-1}(U) \subseteq E$ such that for any fixed $q \in U$, the map $v \to \phi(q, v)$ is linear isomorphism of $\mathbb{R}^r$ onto the fiber E_q. Since the map ϕ is a homeomorphism, we sometimes use ϕ^{-1} but abuse it as ϕ itself.

Due to this local structure, one can feel that there is some resemblance between vector bundles and manifolds, so using this thought we define some of the following things.

Definition 3.12 (Chart). *Let (E, π) be a vector bundle over a topological space X. Then (U, ϕ) is called a chart if U is a trivializing neighbourhood and ϕ is a homeomorphism corresponding to the set U. A chart is also called a trivialization.*

Definition 3.13 (Atlas). *Let (E, π) be a vector bundle over a topological space X. Then $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ is called an atlas if $\{U_\alpha\}$'s are trivializing neighbourhoods and for each α, ϕ_α is a trivialization corresponding to the sets U_α and $\{U_\alpha\}_{\alpha \in I}$ covers X.*

Definition 3.14 (Line bundle). *A vector bundle (E, π) is called a line bundle if the rank of the bundle is one.*

As per the usual practice in mathematics, we will now define morphisms between vector bundles, in particular, when can we say that two vector bundles are equivalent in a certain sense.

Definition 3.15 (Isomorphism of vector bundles). *Two vector bundles $(E_1, \pi_1), (E_2, \pi_2)$ over a same base space X are said to be isomorphic if there exists a map $f : E_1 \to E_2$ such that f is a homeomorphism, $\pi_1 = \pi_2 \circ f$, and f is a linear isomorphism between fibers over p if we fix a point p in X.*

Definition 3.16 (Trivial vector bundle). *A vector bundle of rank r over a space X is called trivial, if $(E, \pi) \cong (X \times \mathbb{R}^r, \tilde{\pi})$ where, $\tilde{\pi}$ is the projection onto the first co-ordinate. We will denote it by ε^r.*

Again, taking motivation from manifold theory, we define the following.

Definition 3.17 (Transition functions). *Let (E, π) be a vector bundle on a topological space X with an open covering $\{U_\alpha\}$ satisfying the condition of local triviality. Now, if $U_\alpha \cap U_\beta$ is non-empty, then we have two homeomorphisms:*
$\phi_\alpha : U_\alpha \cap U_\beta \times \mathbb{R}^r \to \pi^{-1}(U_\alpha \cap U_\beta)$
$\phi_\beta : U_\alpha \cap U_\beta \times \mathbb{R}^r \to \pi^{-1}(U_\alpha \cap U_\beta)$ *And we define the transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{R}^r)$ such that $t_{\alpha\beta}|_x := (\phi_\alpha^{-1} \circ \phi_\beta)_x$.*

Lemma 3.18. *Let (E, π) be a vector bundle over a topological space X. Let $\{U_i\}_{i \in I}$ be trivializing neighbourhoods. Then, the transition functions are continuous.*

Proof. Let $\{\phi_i\}_{i \in I}$ be the corresponding trivializations. Then, we have $(\phi_\alpha^{-1} \circ \phi_\beta)(x, v) = (x, A(x), v)$. Since $\phi_\alpha^{-1}, \phi_\beta$ are continuous, so $(x, v) \mapsto (x, A(x)v)$ is continuous. Hence, it is continuous in both the variables x, and v. Hence, $x \mapsto (x, A(x)v)$ is continuous. Now, since a composition of continuous functions is continuous, hence $x \mapsto (x, A(x)v) \mapsto A(x)v$ is continuous, as pr_2 is a continuous function. Hence, for each fixed $v \in \mathbb{R}^k$, we get $x \mapsto A(x)$, which is continuous. □

Definition 3.19 (Direct sum of vector bundles). *Let (E_1, π_1) and (E_2, π_2) be two vector bundles over a topological space X. Then the direct sum of (E_1, π_1) and (E_2, π_2) is a vector bundle over space X with the total space $E_1 \oplus E_2 := \{(a, b) \in E_1 \times E_2 | E_1(a) = E_2(b)\}$, and the projection $\pi : E_1 \oplus E_2 \to X$, defined as $\pi(a, b) := \pi_1(a)(= \pi_2(b))$ and it is denoted as $(E_1 \oplus E_2, \pi_1 \oplus \pi_2)$.*

Theorem 3.20 (Vector bundle construction theorem). *Let X be topological space. Given an open cover $\{U_i\}_{i \in I}$ of X and a set of continuous functions $t_{ij} : U_i \cap U_j \to GL(r, \mathbb{R}^r)$ defined on each nonempty overlap, such that the cocycle condition $t_{ik}(x) = t_{ij}(x)t_{jk}(x)$ for all $x \in U_i \cap U_j \cap U_k$ holds, then there exists a vector bundle (E, π) of rank r which is trivializable over $\{U_i\}_{i \in I}$ with transition functions t_{ij}.*

Proof. Define $T := \bigsqcup_{i \in I} U_i \times \mathbb{R}^k = \{(i, x, y) | i \in I, x \in U_i, y \in \mathbb{R}^k\}$ (with disjoint union topology and product topology). Define a relation $\sim$ on T as follows:

$$(j, x, y) \sim (i, x, t_{ij}(x)y) \text{ for all } x \in U_i \cap U_j, y \in \mathbb{R}^k.$$

Claim: This relation is an equivalence relation.
$(i, x, y) \sim (i, x, y)$ since $t_{ii}(x) = 1$ for all $i \in I, x \in U_i$ (which is immediate from the cocycle condition). So, we have reflexivity. Similarly, symmetry comes from the fact that $t_{ij}(x) = (t_{ji}(x))^{-1}$ (which is again immediate from the cocycle condition) and for the transitivity part we have

$$(i, x, y) \sim (j, x, t_{ji}(x)y); (j, x, t_{ji}(x)y) \sim (k, x, t_{kj}(x)t_{ji}(x)y) = (k, x, t_{ki}(x)y)$$

by cocycle condition and $(i, x, y) \sim (k, x, t_{ki}(x)y)$ is true by the definition of $\sim$. Hence, the claim is proved. Now, consider $E := T/\sim$ (with the quotient topology) and $\pi : E \to X$ as $\pi([(i, x, y)]) := x$; $\phi_i : U_i \times \mathbb{R}^k \to \pi^{-1}(U_i)$ such that $\phi_i(x, y) := [(i, x, y)]$. The map π is continuous since it is composition of quotient map and projection map, similarly ϕ_i's are also continuous since, they are compositions of inclusion map and quotient map.
Now, it remains to verify that transition functions of this constructed bundle are indeed the t_{ij}'s we started off with. Let $p \in U_i \cap U_j$ be a fixed point then p gets mapped to $t_{ij}(p)$ by the map $\phi_i^{-1} \circ \phi_j$ and we are done. □

Due to this theorem, one can define a vector bundle using the information about the transition functions. We will use this alternative way of defining vector bundles for our further study and denote the corresponding vector bundle as $(\mathcal{U}, g)$ where, $\mathcal{U} = \{U_i\}_{i\in I}$ are trivializing neighbourhoods, and $\{g_{ij}\}_{i,j\in I}$ are the corresponding transition functions. Also, we get a corresponding alternative definition for the vector bundle isomorphism due to the proposition 3.22.

Lemma 3.21. *Let $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over a space X, then there exists an open cover of X which can be treated as trivializing neighbourhood for both E_1 and E_2.*

Proof. Let $\{(U_i, \phi_i)_{i\in I}\}, \{(V_j, \psi_j)_{j\in J}\}$ be atlases of E_1, E_2 respectively. Then we can consider the open sets of the form $U_i \cap V_j$ for $i \in I, j \in J$. Note that this covers the whole space. Now, define new trivializations $\tilde{\phi}_{ij} := \phi_i|_{U_i\cap V_j}$ and $\tilde{\psi}_j := \psi_j|_{U_i\cap V_j}$. Note that this argument can be extended to a collection of finitely many vector bundles over a same base space. □

Proposition 3.22. *Let $(E, \pi), (E', \pi')$ be two vector bundles of rank r over same base space X, also, with same trivializing neighbourhoods $\{U_i\}_{i\in I}$ but with transition functions t_{ij}, t'_{ij}. Then E and E' are isomorphic if and only if there exist continuous fuctions $t_i : U_i \to GL_(r, \mathbb{R})$ such that $t'_{ij}(x) = t_i^{-1}(x)t_{ij}(x)t_j(x)$ for all $x \in U_i \cap U_j$, for all $i, j \in I$.*

Proof. ($\Leftarrow$) To define a map $f : E \to E'$ which will be homeomorphism and linear isomorphism for fixed $p \in X$, define $f([(i, x, v)]) := [(i, x, t_i(x)v)]$. To check that it is well-defined, consider another representative from the same class, say $(j, x, t_{ji}(x)v)$. Then

$$f([(j, x, t_{ji}(x)v)]) = [(j, x, t_j(x)t_{ji}(x)v)] = [(i, x, t'_{ij}(x)t_j(x)t_{ji}(x)v)]$$

and we have:

$$[(i, x, t'_{ij}(x)t_j(x)t_{ji}(x)v)] = [(i, x, t_i(x)t_{ij}(x)t_{ji}(x)v)] = [(i, x, t_i(x)v)].$$

Hence, it is well-defined. It is homeomorphic due to construction and for fixed $p \in X$. Moreover, v gets mapped to t_iv which is just scaling by an invertible matrix, hence, it is a linear isomorphism after fixing a point.
($\Rightarrow$) Let us define $t_i : U_i \to GL(r, \mathbb{R})$ as follows:

$$t_i(x) := (v \xrightarrow{i} (x, v) \xrightarrow{k} (x, t_i(x)v) \to t_i(x)v)$$

Here, k denotes $\phi_i'^{-1} \circ f \circ \phi_i$. This map is clearly linear and invertible. Also, it satisfies the equation: $t_j(x)t_{ji}(x) = t'_{ji}(x)t_i(x)$ since, $t_{ij} = \phi_i^{-1}\phi_j$. Now, to prove that t_i's are continuous, we can use the similar method as we did for proving lemma 3.18. The only difference is $(\phi_\alpha^{-1} \circ \phi_\beta)$ will be replaced by $(\phi_i'^{-1} \circ f \circ \phi_i)$ which can still be used since ϕ_i, ϕ_i' and f are homeomorphisms. □

Definition 3.23 (Whitney sum of vector bundles). *Let (E_1, g) and (E_2, g') be two vector bundles of rank k, l respectively over a topological space X. Let $\{U_i\}_{i\in I}$ be a common refinement of trivializing neighbourhoods which cover X. Then the Whitney sum of (E_1, g) and (E_2, g') is a vector bundle over space X defined using transition data as follows:*

$$(g \oplus g')_{ij} : U_i \cap U_j \to GL_{k+l}(\mathbb{R})$$

$$(g \oplus g')_{ij} := \begin{bmatrix} g_{ij}(x) & 0 \\ 0 & g'_{ij}(x) \end{bmatrix}$$

That is, fiber over each $x \in X$ in the Whitney sum is the direct sum of fibers over x in the vector bundles in the summand. Denote this by $E_1 \oplus_W E_2$.

Proposition 3.24. *Let $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over a space X, then the direct sum $E_1 \oplus E_2 \cong E_1 \oplus_W E_2$.*

Proof. $E_1 \oplus_W E_2 \subseteq E_1 \oplus E_2$ is trivial. For the other side, let $(e_1, e_2) \in E_1 \oplus E_2$, that is $\pi_1(e_1) = \pi_2(e_2) = x$ (say). We can write $(e_1, e_2) = (e_1, 0_x) + (0_x, e_2)$,that is, as sum of two elements which come from the sets which are isomorphic to E_1 and E_2. Where 0_x is the zero vector in the fiber $(E_i)_x$ $i = 1, 2$. Moreover, intersection of these two sets is also zero. Hence the proof is complete. So, from now on, we will write $E_1 \oplus E_2$ for both direct sum as well as Whitney sum of E_1 and E_2. □

Definition 3.25 (Refinement). *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Then $\mathcal{V} = \{V_j\}_{j\in J}$ is called a refinement of $\mathcal{U}$ if $\mathcal{V}$ is an open cover of X, and for each $j \in J$, there exists an $i \in I$ such that $V_j \subseteq U_i$.*

Definition 3.26 (Locally finite collection). *A collection $\{U_i\}_{i\in I}$ of subsets of a topological space X is said to be locally finite, if for each point $x \in X$, there exists an open neighbourhood U_x of x such that $\{i \in I | U_x \cap U_i \neq \varnothing\}$ is a finite set.*

Definition 3.27 (Paracompact space). *A topological space X is said to be paracompact, if its any open cover has a refinement, which is locally finite.*

Remark 3.28. *If X is a paracompact topological space, then its any open cover (say, $\{U_\alpha\}_{\alpha\in I}$) has a refinement (say, $\{V_\beta\}_{\beta\in J}$), which is locally finite in a stronger sense, meaning for each point $x \in X$ there exists an open set $V_x \in \{V_\beta\}_{\beta\in J}$ such that $\{\beta \in J | V_x \cap V_\beta \neq \varnothing\}$ is a finite set. In order to see a proof of this, let $\{U_\alpha\}_{\alpha\in I}$ be an open cover. Now, due to the paracompactness there exists a locally finite refinement, say, $\{W_\zeta\}_{\zeta\in K}$. Now, due to local finiteness, for each $x \in X$, there exists an open neighbourhood U_x of x such that $\{\zeta \in K | U_x \cap W_\zeta \neq \varnothing\}$ is a finite set. Now, define $V_{x,\zeta} := U_x \cap W_\zeta$ for all $x \in X, \zeta \in K$. So, it is clear from the construction that the refinement $\{V_{x,\zeta}\}_{x\in X,\zeta\in K}$ of $\{U_\alpha\}_{\alpha\in I}$* is *indeed locally finite in a stronger sense.*

Definition 3.29. *Let $f \in C(X)$. The support of $f := \overline{f^{-1}(\mathbb{R} - \{0\})}$. It is denoted by $Supp(f)$.*

Definition 3.30 (Partition of unity). *Let X be a topological space and $\{U_\alpha\}_{\alpha \in J}$ be an open cover of X. Then a partition of unity is a collection of continuous functions $\{\phi_\alpha\}_{\alpha \in J} \subseteq C(X, [0,1])$ such that $supp(\phi_\alpha) \subseteq U_\alpha$ for all $\alpha \in J$, for each point $x \in X$, $\phi_\alpha(x)$ is non-zero for at most finitely many α's in J, $\sum_{\alpha \in J} \phi_\alpha(x) = 1$ for all $x \in X$.*

Lemma 3.31. *Let X be a topological space, let $\{U_i\}_{i \in I}$ be an open cover, and let $(\phi : J \to I), \{V_j\}_{j \in J}$, be a refinement to a locally finite cover. Then $\{W_i\}_{i \in I}$ with $W_i := \bigcup_{j \in \phi^{-1}(i)} V_j$ is still a locally finite refinement of $\{U_i\}_{i \in I}$.*

Proof. It is clear by construction that $W_i \subseteq U_i$ for each $i \in I$, hence we have a refinement. So, we need to show local finiteness. Consider a point $x \in X$. By assumption, $\{V_j\}_{j \in J}$ is locally finite, hence there exists a neighbourhood U_x of x and a finite set $K \subseteq J$ such that $U_x \cap V_j = \varnothing$ for all $j \in J - K$. Hence, by construction, $U_x \cap W_i = \varnothing$ for all $i \in I - \phi(K)$. Since $\phi(K) \subseteq I$ is also a finite set we get that $\{W_i\}_{i \in I}$ is locally finite. □

Lemma 3.32. *Let X be a normal space, A be a closed set contained in an open set U. Then there exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$.*

Proof. Consider two disjoint closed subsets A and $B = U^C$. Since X is normal there exist disjoint open sets V, W containing A, B respectively. So, it is sufficient to prove that $\overline{V} \subseteq U$. We know that, by definition, $\overline{V} = \bigcap_{C \text{ is closed}, V \subseteq C} C$. So, since $V \subseteq W^C$ and W^C is closed as W is open, we get $\overline{V} \subseteq W^C$, and we have $U^C = B \subseteq W$ which implies $W^C \subseteq (U^C)^C = U$ which in turn implies $\overline{V} \subseteq W^C \subseteq U$. □

Lemma 3.33 (Shrinking lemma). *Let X be a normal space and let $\{U_i\}_{i \in I}$ be a locally finite open cover. Then there exists another open cover $\{V_i\}_{i \in I}$ such that $V_i \subseteq \overline{V_i} \subseteq U_i$ for all $i \in I$.*

Proof. Consider an $\alpha \in I$. Denote $A_\alpha := X - \bigcup_{i \in I,\, i \neq \alpha} U_i$. This is a closed subset of X, moreover, $A_\alpha \subseteq U_\alpha$. Hence, using the lemma 3.32 there exists an open set V_α containing A_α such that $\overline{V_\alpha} \subseteq U_\alpha$. Since α was chosen arbitrarily, we get the result. □

Lemma 3.34 (Urysohn's lemma). *A topological space X is normal if and only if for any two given disjoint closed sets $A, B \subseteq X$, there exists $f \in C(X, [0,1])$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Proposition 3.35. *Let X be a paracompact Hausdorff space. Then for every open cover $\{A_i\}_{i \in I}$ there is a subordinate partition of unity.*

Proof. Let $\{U_i\}_{i\in I}$ be a locally finite refinement of $\{A_i\}_{i\in I}$ which exists due to paracompactness of X moreover, by lemma 3.31 we may assume that this has same index set. It is now sufficient to show that this locally finite cover $\{U_i\}_{i\in I}$ admits a subordinate partition of unity, since this will then also be subordinate to the original cover. Since paracompact Hausdorff spaces are normal by lemma C.4, we may apply lemma 3.33 to the given locally finite open cover $\{U_i\}_{i\in I}$, to obtain a smaller locally finite open cover $\{V_i\}_{i\in I}$, and then apply the lemma once more to that result to get a yet small open cover $\{W_i\}_{i\in I}$, so that now $W_i \subseteq \overline{W_i} \subseteq V_i \subseteq \overline{V_i} \subseteq U_i$ for all $i \in I$. It follows that for each $i \in I$, we have two disjoint closed subsets, namely the $\overline{W_i}$ and $X - V_i$. Now since paracompact Hausdorff spaces are normal, Urysohn's lemma says that there exist continuous functions $h_i : X \to [0,1]$ with the property that $h_i(\overline{W_i}) = \{1\}, h_i(X - V_i) = \{0\}$. In particular, $h_i^{-1}((0,1]) \subseteq V_i$ and hence that $Supp(h_i) = \overline{h_i^{-1}(0,1]} \subseteq \overline{V_i} \subseteq U_i$. It just remains to normalize these functions so that they indeed sum to unity. So, consider the continuous function $h : X \to [0,1]$ defined on $x \in X$ by $h(x) := \sum_{i\in I} h_i(x)$. Notice that the sum on the right has only a finite number of non-zero summands, due to the local finiteness of the cover, so this is well-defined. Moreover, notice that $h(x) \neq 0$ because $\{\overline{W_i}\}_{i\in I}$ is a cover so that there is $i_x \in I$ with $x \in \overline{W_{i_x}}$, and since $h_i(\overline{W_{i_x}}) = \{1\}$, by the above. Hence it makes sense to define $f_i := \frac{h_i}{h}$. This now implies $\sum_{i\in I} f_i \equiv 1$, and so, $\{f_i\}_{i\in I}$ is a partition of unity as required. □

Definition 3.36. *Let $(\mathcal{U}, g), (\mathcal{U}, g')$ be two vector bundles over a same space X with transition functions over a common refinement $\mathcal{U} = \{U_i\}_{i\in I}$. Then we define tensor product of E_1 and E_2 and denote it by $E_1 \otimes E_2$ as $(\mathcal{U}, g \otimes g')$.*

For this definition to make sense, one needs to check that the transition functions $g \otimes g'$ satisfy the cocycle data. But this is evident since, for each $x \in X$

$$(g_{ij}(x) \otimes g'_{ij}(x)) \cdot (g_{jk}(x) \otimes g'_{jk}(x)) = (g_{ij}(x) \cdot g'_{ij}(x) \otimes (g_{jk}(x) \cdot g'_{jk}(x))) = g_{ik}(x) \otimes g'_{ik}(x)$$

Definition 3.37 (Euclidean structure). *Let (E, π) be a vector bundle over a topological space X. A Euclidean structure on the vector bundle is a bundle map $g : E \otimes E \to (\varepsilon^1, pr_1)$ such that for each $x \in X$, we get an inner product over the vector space E_x when we restrict the map g to $(E_x \otimes E_x)$.*

Remark 3.38. *If (E, π) is a vector bundle of rank k over a space X with the trivializing neighbourhoods $\{U_i\}_{i\in I}$. Then a Euclidean structure on E is equivalent to a collection of maps $A_i = \langle \cdot, \cdot \rangle_{U_i} : U_i \to M_k(\mathbb{R})$ such that $\langle \cdot, \cdot \rangle_x$ is a positive definite symmetric matrix and for $x \in U_i$, $\langle v, w \rangle_x := \langle \phi_i^{-1}(x, v), \phi_i^{-1}(x, w) \rangle_g$ satisfying*

$$\langle v, w \rangle_{U_j} \equiv \langle g_{ij}(\cdot)v, g_{ij}(\cdot)w \rangle_{U_i} \text{ for } x \in U_i \cap U_j.$$

Remark 3.39. *If (E, π) is a trivial vector bundle of rank k over a space X then there exists a Euclidean structure on E. Since, there is a global chart namely, the set whole set X itself.*

So, we can define an inner product $\langle v, w\rangle_x := \langle pr_2(x, v), pr_2(x, w)\rangle_{\mathbb{R}^k}$ for all $x \in X$. Note that this satisfies the required conditions mentioned in the remark 3.38 as $\phi^{-1} = pr_2$ for trivial vector bundles, and there is only one transition function and that too is equal to identity matrix.

Proposition 3.40. *Let (E, π) be a vector bundle of rank k over a topological space X. If X is paracompact and Hausdorff, then we get a Euclidean structure on (E, π).*

Proof. Let $\{(U_i, \phi_i)\}_{i\in I}$ be an atlas of E and $\{\rho_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. For each $i \in I$, define $A_i : U_i \to M_k(\mathbb{R})$ as $\langle v, w\rangle_x := \langle pr_2(x, v), pr_2(x, w)\rangle_{\mathbb{R}^k}$. Now, extend A_i's to $\tilde{A}_i$'s by multiplying it by ρ_i. Now we will construct a Euclidean structure on E as follows: For each $x \in X$ there exists an open neighbourhood V_x and a finite subset $J_x \subseteq I$ such that V_x does not intersect with open sets in the collection $\{U_i\}_{i\in I-J_x}$. Hence we can define $\sum_{i\in I}(\rho_i \cdot A_i)$ as a map from V_x to $M_k(\mathbb{R})$ which is positive definite symmetric matrix for each x since it is a linear combination of finitely many inner products with non-negative coefficients which do not vanish at the same time. Now, it remains to prove that this function satisfies

$$\langle v, w\rangle_{U_j} \equiv \langle g_{ij}(\cdot)v, g_{ij}(\cdot)w\rangle_{U_i} \text{ for } x \in U_i \cap U_j.$$

We can write $g_{ij}(x) = (\phi_j \cdot \phi_i^{-1})_x$. Hence it remains to check that the following is true.

$$\langle v, w\rangle_{U_j} \equiv \langle(\phi_j \cdot \phi_i^{-1})(x, v), (\phi_j \cdot \phi_i^{-1})(x, w)\rangle_{U_i} \text{ for } x \in U_i \cap U_j.$$

Now, using bilinearity of the inner product, it remains to prove that

$$\langle\phi_j^{-1}(x, v), \phi_j^{-1}(x, w)\rangle_{U_j} \equiv \langle\phi_i^{-1}(x, v), \phi_i^{-1}(x, w)\rangle_{U_i} for\ x \in U_i \cap U_j.$$

But this is true since $\phi_i^{-1}(x, v) = \phi_j^{-1}(x, v)$ for $x \in U_i \cap U_j$ by the definition of E being a vector bundle. Hence we are done. □

Proposition 3.41. *For any given line bundle (E, g_{ij}) over a paracompact topological space X, there exists a line bundle (E', g'_{ij}) such that $g'_{ij} \in \{1, -1\}$, E and E' are isomorphic.*

Proof. From proposition 3.40 we know that there exists a Euclidean structure on E. Let $\{U_i\}_{i\in I}$ be the trivializing neighbourhood of E. So, the remark 3.38 implies that there exists continuous functions $A_i : U_i \to M_1(\mathbb{R})(= \mathbb{R})$ such that $A_i(x)$ is a positive definite symmetric matrix. Hence, we can use Schur's decomposition [23] for real matrices, which implies $A_i(x)$ can be written as $A_i(x) = U(x)D(x)U^t(x)$, where $U(x)$ is an orthogonal matrix (that is $U^{-1}(x) = U^t(x)$) and $D(x)$ is a diagonal matrix since $A(x)$ is symmetric. Moreover, The diagonal elements of $D(x)$ are the eigenvalues of matrix $A(x)$ hence, they are positive, since $A(x)$ is positive definite. Let $\sqrt{A_i(x)}$ denote the positive square root of $A_i(x)$ (entrywise). From the remark 3.38 we also have that $v^tA_j(x)w = (g_{ij}(x)v)^tA_i(x)g_{ij}(x)w$, for all $v, w \in \mathbb{R}, x \in U_i \cap U_j$. This implies $A_j(x) = g_{ij}^t(x)A_i(x)g_{ij}(x)$. Now using the symmetry of the matrices, we can write it as

$$\sqrt{A_j^t}\sqrt{A_j} = g_{ij}^t(x)\sqrt{A_i^t(x)}\sqrt{A_i^t(x)}g_{ij}(x)$$

After rearranging the terms we get,

$$\left(\left(\sqrt{A_j^t}\right)^{-1} g_{ij}^t \sqrt{A_i^t}\right)\left(\sqrt{A_i} g_{ij}\left(\sqrt{A_j}\right)^{-1}\right) = Id$$

which implies $\left(\sqrt{A_i} g_{ij}\left(\sqrt{A_j}\right)^{-1}\right)$ is orthogonal. It was sufficient to find $h_i : U_i \to GL_1(\mathbb{R})(= \mathbb{R} - \{0\})$ such that $h_i \cdot g_{ij} = \tilde{g}_{ij} \cdot h_j$ on $U_i \cap U_j$, where $\tilde{g}_{ij} \in \{1, -1\}$. So, $h_i = \sqrt{A_i}$ would work, simply because $h_i(x) \cdot g_{ij}(x) \cdot (h_j(x))^{-1} = \sqrt{A_i(x)} \cdot g_{ij}(x) \cdot \sqrt{A_j(x)} \in \{1, -1\}$. This completes the proof. □

Remark 3.42. *One can easily observe that the same procedure can be used to show that when X is paracompact and a vector bundle E over X is of rank k, then we can find an isomorphic vector bundle whose transition data lie in the group $O_k(\mathbb{R})$.*

Proposition 3.43. *Let (E, t_{ij}), (E', t'_{ij}) be two line bundles over a paracompact topological space X with common trivializing neighbourhoods $\{U_i\}_{i \in I}$ which are isomorphic with $t_{ij}(x), t_{ij}(x)' \in \{1, -1\}$ for all $x \in U_i \cap U_j$ for all $i, j \in I$ then we can have corresponding functions, t_i's such that $t_i(x) \in \{1, -1\}$ for all $x \in U_i$ for all $i \in I$.*

Proof. Using the proposition 3.41, and proposition 3.22 we can say that there exist continuous functions $h_i : U_i \to GL_1(\mathbb{R})$ for every $i \in I$ such that $h_i(x)t'_{ij}(x) = t_{ij}(x)h_j(x)$ for all $x \in U_i \cap U_j$, where $t_{ij}, t'_{ij} \in \{1, -1\}$. Hence, taking absolute value on both the sides, we get $|h_i(x)| = |h_j(x)|$ for $x \in U_i \cap U_j$. Now define $t_i(x) := \dfrac{h_i(x)}{|h_i(x)|}$ for all $x \in U_i$. Note that this makes sense simply because h_i's are non-zero, as they belong to $GL_1(\mathbb{R})$. Moreover, $t_i(x) \in \{1, -1\}$ for all $x \in U_i$, for all $i \in I$ which also satisfy the equation $t_i(x)t'_{ij}(x) = t_{ij}(x)t_j(x)$ for all $x \in U_i \cap U_j$. Hence we are done. □

3.2 Galois Correspondence

We will refer [9] for this entire subsection. Now, let us first construct an index two subgroup from a double cover. For this, note that we have a homomorphism induced by the covering map (which is valid for any continuous map, not just covering) $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ such that $p_*([\gamma]) := [p\gamma]$. This implies $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a subgroup of $\pi_1(X, x_0)$.

Proposition 3.44. *The number of sheets of a covering space p: $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.*

Proof. Let hg be an element in the right coset of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ $(=: H)$ with respect to a loop g in X based at x_0. Consider the lift of hg, which is $\tilde{h}\tilde{g}$. $\tilde{h}\tilde{g}$ ends at the same point where $\tilde{g}$ ends, and does not depend on $\tilde{h}$ since it is a loop in X̃ (based at $\tilde{x}_0$). Now, define a map $\phi : \{Hg | g \in \pi_1(X, x_0)\} \to p^{-1}(x_0)$ such that $\phi(Hg) := \tilde{g}(1)$. The path-connectedness of $\tilde{X}$ implies that ϕ is surjective. Since, $\tilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ say, $\tilde{x}_1$ by a path say, $\tilde{g}$ which can be projected using covering map p to a loop say, g at x_0. So that $\phi(Hg) = \tilde{x}_1$. Now, to show that ϕ is injective, let $\phi(Hg_1) = \phi(Hg_2)$ which implies that $g_1\bar{g}_2$ lifts to a loop $\tilde{g}_1\tilde{\bar{g}}_2$ in $\tilde{X}$ based at $\tilde{x}_0$. So, $g_1 g_2^{-1} \in H \Rightarrow Hg_1 = Hg_2$. $\square$

From the Proposition 3.22 we can say that, if it is a double cover, then the subgroup will be of index two. This gives us a way of getting an index two subgroup starting from a double cover and now we will see a method to get a double cover from an index two subgroup of the fundamental group of a topological space with certain constraints.

Theorem 3.45. *Let X be path-connected, locally path-connected, semi-locally simply connected space. Then there exists a simply connected covering space of X (say $\tilde{X}$). (That is, $\pi_1(\tilde{X}) = 0$)*

Proof. Define $\tilde{X} := \{[\gamma] | \gamma \text{ is a path in } X \text{ starting at } x_0\}$ where, [.] denotes homotopy class with fixed endpoints. Also, define $p : \tilde{X} \to X$ such that $p([\gamma]) := \gamma(1)$. This is well defined, since, endpoints are fixed and is surjective due to path-connectedness. Let $\mathcal{U}$ be the collection of path-connected open sets $U \subseteq X$ such that $\pi_1(U) \to \pi_1(X)$ is trivial. If $\pi_1(U) \to \pi_1(X)$ is trivial for one choice of basepoint in U, then so is for all choices of basepoints due to path-connectedness of U. A path-connected open subset $V \subseteq U$ is also in $\mathcal{U}$ since the composition
$\pi_1(V) \overset{i}{\hookrightarrow} \pi_1(U) \to \pi_1(X)$ will also be trivial. Let $x \in X$ be a point. Then there exists an open set U_x which is simply connected, due to semilocal simply connectedness. Now find a subset of U_x say, V_x which is path-connected. This is possible due to local path-connectedness. Hence, the sets in the collection $\mathcal{U}$ covers X. Using similar argument we can show that given $U_1, U_2 \in \mathcal{U}$ and $p \in U_1 \cap U_2$ there exists $U_3 \in \mathcal{U}$ such that $p \in U_3 \subseteq U_1 \cap U_2$. This implies $\mathcal{U}$ is basis for the topology on X.
Now, given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, define:

$$U_{[\gamma]} := \{[\gamma\eta] | \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}$$

Property: $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$. Proof of the property goes as follows: $[\gamma'] \in U_\gamma$ which implies $\gamma' = \gamma\eta$ for some path η in U. Then the elements of $U_{[\gamma']}$ will look like $[\gamma\eta\mu]$, which implies $U_{[\gamma']} \subseteq U_{[\gamma]}$ and elements of $U_{[\gamma]}$ can be written as $[\gamma\mu] = [\gamma\eta\overline{\eta}\mu] = [\gamma'\overline{\eta}\mu]$, which implies $U_{[\gamma]} \subseteq U_{[\gamma']}$.
We can define topology on $\tilde{X}$ by calling $U_{[\gamma]}$'s as basis elements. Since, they cover $\tilde{X}$ when we consider $U = X$ and γ as a constant loop and finite intersection of these sets is also open. To see this, consider two such sets $U_{[\gamma]}$ and $V_{[\gamma']}$. We will show that for any element $[\gamma''] \in U_{[\gamma]} \cap V_{\gamma'}$ there exists an open set $W_{[\gamma'']}$ which lies totally inside the intersection. Due to the previous property $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$ implies $U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$. Since $\mathcal{U}$ is a basis, there exists $W \in \mathcal{U}$ such that $\gamma''(1) \in W \subseteq U \cap V$, which implies $[\gamma''] \in W_{[\gamma'']} \subseteq U_{[\gamma]} \cap V_{[\gamma']}$. Now we claim that $(\tilde{X}, p)$ is indeed a covering with $p^{-1}(U) = \bigsqcup\limits_{\alpha \in \pi_1(X)} U_{[\alpha\gamma]}$ for any fixed path γ in

X, that is, $\tilde{X}$ forms a covering space of degree $|\pi_1(X)|$. In order to prove this claim, it is clear from the definition of p that $p^{-1}(U) = \bigcup_{\zeta(1)\in U} U_{[\zeta]}$. Now, we will show that $U_{[\zeta]} = U_{[\alpha\gamma]}$ for some $\alpha \in \pi_1(X)$ for a fixed γ such that $\gamma(1) \in U$. Note that end point at time 1 of ζ and γ does not matter. (Refer figure 4).

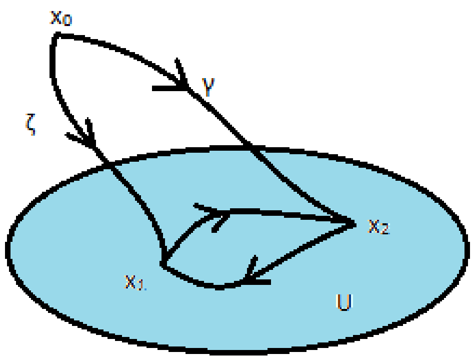


Figure 4: Use of path-connectedness of U.

Since there is a path say, τ_1 from $\gamma(1)$ to $\zeta(1)$ due to path-connectedness of U and similarly a path say, τ_2 from $\zeta(1)$ to $\gamma(1)$ and $\tau_1\overline{\tau_2}$ is nullhomotopic to a point in X since, U is simply connected. So, consider the situation when $\zeta(1) = \gamma(1)$. (Refer figure 5).

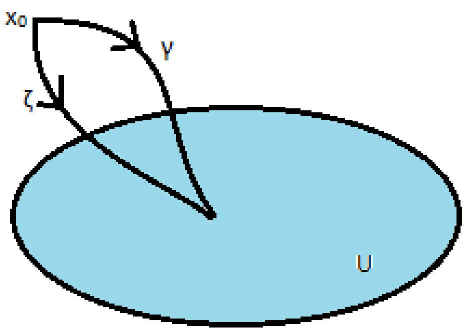


Figure 5: Use of simply-connectedness of U.

Then $\alpha = \zeta\overline{\gamma}$ serves the purpose and to show that $U_{[\alpha_1\gamma]} \cap U_{[\alpha_2\gamma]}$ is empty, let us assume the contrary. Let $[\beta] \in U_{[\alpha_1\gamma]} \cap U_{[\alpha_2\gamma]}$ then, $\beta \sim \alpha_1\gamma\eta_1 \sim \alpha_2\gamma\eta_2$, hence $[\alpha_1\gamma\eta_1\overline{\eta_2\gamma\alpha_2}] = 0$ and we have $[\eta_1\overline{\eta_2}] = 0$ due to simply connectedness of U and hence a contradiction.
Now let us show that $\tilde{X}$ is path-connected. For a given $[\gamma] \in \tilde{X}$, define $f_\gamma : [0,1] \to \tilde{X}$ such that $f_\gamma(t_0) := [\gamma(t_0\cdot)](=: [\gamma_{t_0}])$. Note that $f_\gamma(0) = [x_0]$ (constant loop); $f_\gamma(1) =$

$[\gamma]$. Now, to show that f_γ is continuous consider $U_{[\alpha]}$ as an open set in $\tilde{X}$. If $f_\gamma^{-1}(U_\gamma)$ is empty, then we are done. Otherwise, (Refer figure 6).

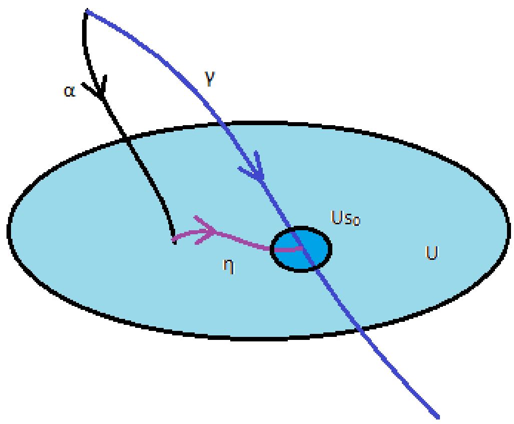


Figure 6: Continuity of f_γ.

$f_\gamma^{-1}(U_{[\alpha]}) = \{s \in [0,1] | \gamma(s.t) \sim \alpha\eta(t) \text{ for some } \eta\}$. Let, $s_0 \in f_\gamma^{-1}(U_{[\alpha]})$ which implies at $t = 1$: $\gamma(s_0) \in U$; now, since γ is continuous and U is open, $\gamma^{-1}(U)$ is also open and contains s_0. Call it U_{s_0}. Hence, f_γ is continuous and hence $\tilde{X}$ is path-connected. Now, to show that $\tilde{X}$ is simply connected, it is enough to show that p_* is trivial, since, p_* is injective (by Proposition 3.10). An element in the image of p_* is γ based at x_0 such that when lifted to $\tilde{X}$, we get a loop in $\tilde{X}$ based at $[x_0]$. So, the path $t \to [\gamma_t]$ is a loop, then $[x_0] = [\gamma]$. □

Now, if we are given a subgroup H of $\pi_1(X, x_0)$, then we define a relation on elements of $\tilde{X}$ as follows: $[\gamma] \sim [\gamma']$ if and only if $[\gamma\overline{\gamma'}] \in H$. This is an equivalence relation, since H is a subgroup. (Reflexivity comes from existence of identity, symmetry comes from existence of inverses, and transitivity comes from closure under group operation). So, we define $X_H := \tilde{X}/\sim$. We can show (in similar way, as shown for $\tilde{X}$), that $q : \tilde{X} \to X_H$ is a covering map with $q^{-1}(U_{[\alpha]}) = \bigsqcup_{\alpha \in H} U_{[\alpha\delta]}$. Hence, degree of q is $|H|$. Similarly, we also get $p_H : X_H \to X$ which is cover of degree $\dfrac{|\pi_1(X)|}{|H|}$ = Index of H in $\pi_1(X)$.

Proposition 3.46. *If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Proof. Suppose $\tilde{x}_1$ is another basepoint in $p^{-1}(x_0)$. Let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Then, $\tilde{\gamma}$ projects a loop γ in X representing some element $g \in \pi_1(X, x_0)$. Set $H_i = p_*(\pi_1(\tilde{X}, \tilde{x}_i))$ for $i = 0, 1$. Then $g^{-1}H_0 g \subseteq H_1$, since for $\tilde{f}$ a loop at $\tilde{x}_0$, $\overline{\tilde{\gamma}}\tilde{f}\tilde{\gamma}$ is a loop at $\tilde{x}_1$, similarly, $gH_1g^{-1} \subseteq H_0$. This implies $H_1 = g^{-1}H_0 g$. Conversely, if we have

$H_1 = g^{-1}H_0 g$, choose γ representing g. Lift γ to a path $\tilde{\gamma}$ at $\tilde{x}_0$ and let $\tilde{x}_1 := \tilde{\gamma}_1$, which implies $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H_1$. So, in our case, subgroups are of index two and hence normal by Proposition 3.2, so they will not get affected by conjugacy classes. □

Now, we will show that the developed correspondence agrees over isomorphism classes of the mentioned objects.

Proposition 3.47. *If X is path-connected and locally path-connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic if and only if*

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2)).$$

Proof. ($\Rightarrow$) $p_1 = p_2 \circ f$, which implies, $p_{1*} = p_{2*} \circ f$, where, f_* is isomorphism, since f is homeomrphism.
($\Leftarrow$) By using lifting criterion (Proposition 2.3), lift p_1 to $\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ with $p_2 \circ \tilde{p}_1 = p_1$ and symmetrically, $\tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \to (\tilde{X}_1, \tilde{x}_1)$ with $p_1 \circ \tilde{p}_2 = p_2$. Then by unique lifting property (Proposition 3.9), we get $\tilde{p}_1 \circ \tilde{p}_2 = id$, and $\tilde{p}_2 \circ \tilde{p}_1 = id$, which implies, $\tilde{p}_1$ and $\tilde{p}_2$ are inverse isomorphisms. □

3.3 Correspondence between line bundles and double covers

Let us first consider a line bundle (E, π) over a paracompact topological space X and try to give a corresponding double cover over the space X. Now, by the Proposition 3.41 we can consider an isomorphic line bundle whose transition function belongs to $\{1, -1\}$ and from the given data of transition functions we can build the total space E, as we did in the proof of vector bundle construction theorem (3.20). Let $\{(U_i, \phi_i)\}_{i \in I}$ be an atlas of space X, then we define a space

$$\tilde{X} := \{[(i, x, k)] | i \in I, x \in U_i, k = 1 or -1\} (\subseteq E)$$

with the subspace topology. Note that this is a well-defined space, since $t_{ij}(x)$'s $\in \{1, -1\}$ by proposition 3.41. Now, define $p : \tilde{X} \to X$ such that $p([(i, x, k)]) := x$. Note that this is indeed well-defined and the trivializing neighbourhoods would work also as evenly covered neighbourhoods since, for each $x \in X$, there exists a U_i for some $i \in I$ such that $x \in U_i$, and

$$p^{-1}(U_i) = \{[(i, x, 1)] | x \in (U_i)\} \sqcup \{[(i, x, -1)] | x \in U_i\}.$$

Moreover, $p^{-1}(x) = [(j, x, k)] = [(i, x, t_{ij}(x)k)]$, since $k, t_{ij}(x) \in \{1, -1\}$ so, $t_{ij}(x) \cdot k \in \{1, -1\}$, hence it is indeed a double cover.

Now, given a double cover, let us give a line bundle. Let $\{U_\alpha\}_{\alpha \in I}$ be a evenly covered neighbourhood of the given double cover. Consider $U_i, U_j \in \{U_\alpha\}$ and $p^{-1}(U_i) = V_{i1} \sqcup V_{i2}$ and $p^{-1}(U_j = V_{j1} \sqcup V_{j2})$. Then

$$p^{-1}(U_i \cap U_j) = (V_{i1} \cap V_{j1}) \sqcup (V_{i1} \cap V_{j2}) \sqcup (V_{i2} \cap V_{j1}) \sqcup (V_{i2} \cap V_{j2}).$$

So, for any point $x \in U_i \cap U_j$, we will have $p^{-1}(x) = \{y, z\}$ such that one of y, z will lie in V_{i1}, call it $\tilde{x}_1$ (1), and the other point to be $\tilde{x}_2$. Now, $\tilde{x}_1$ can lie in V_{j1}, or V_{j2} and $\tilde{x}_2$ will lie in V_{j2}, or V_{j1} respectively, since

$$p^{-1}(U_i \cap U_j) = (V_{i1} \cap V_{j1}) \sqcup (V_{i1} \cap V_{j2}) \sqcup (V_{i2} \cap V_{j1}) \sqcup (V_{i2} \cap V_{j2}).$$

So, we define the transition function as follows.

$$t_{ij}(x) = \begin{cases} 1, & \text{if } p^{-1}(x) \in (V_{i1} \cap V_{j1}) \sqcup (V_{i2} \cap V_{j2}) \\ \text{-1} & \text{if } p^{-1}(x) \in (V_{i1} \cap V_{j2}) \sqcup (V_{i2} \cap V_{j1}) \end{cases}$$

Note that we made a choice of index to be i at (1), but it can be easily checked that we will get the same t_{ij} if we choose j, instead of i. In order to check the continuity of t_{ij}, note that the codomain is $\{1, -1\}$ with discrete topology. So, let us look at $t_{ij}^{-1}(1)$ and $t_{ij}^{-1}(-1)$.

$$t_{ij}^{-1}(1) = p(V_{i1} \cap V_{j1})(= V_{i1} \cap V_{j2}), \text{ and } t_{ij}^{-1}(-1) = p(V_{i1} \cap V_{j2})(= V_{i2} \cap V_{j1})$$

Now, since $V_{i1}, V_{i2}, V_{j1}, V_{j2}$ are open, and so is their intersection, and p is a homeomorphism, hence t_{ij}'s are continuous. It can be checked using case by case analysis that they satisfy the cocycle condition. For example, if $x \in U_i \cap U_j \cap U_k$, and $t_{ij}(x) = t_{jk} = -1$ i.e. $\tilde{x}_1 \in V_{i1} \cap V_{j2}$, and using symmetry for the second equation, we get $t_{kj} = -1$, hence $\tilde{x}_1 \in V_{k1} \cap V_{j2}$, that is, $\tilde{x}_1 \in V_{i1} \cap V_{k1}$, that is, $t_{ik}(x) = 1$, which proves the cocycle condition for this case. So, using the same evenly covered neighbourhoods as trivializing neighbourhoods, we get a line bundle, as required. Now, we will show that the developed correspondence agrees over isomorphism classes of the mentioned objects.

Proposition 3.48. *Let X be a topological space, $(E_1, \pi_1), (E_2, \pi_2)$ be two line bundles, $\tilde{X}_1, \tilde{X}_2$ be the corresponding double covers. Then E_1 and E_2 are isomorphic if and only if $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic.*

Proof. Let $E_1 \cong E_2$, be two isomorphic line bundles with transition functions belonging to $\{1, -1\}$. Now, define $f : \tilde{X}_1 \to \tilde{X}_2$ such that

$$f([i, x, k]) := [(i, x, t_i(x)k)]$$

Note that $[(i, x, t_i(x)k)] \in \tilde{X}_2$ due to Proposition 3.43. To check well-definedness, see that

$$f([(j, x, t_{ji}(x)k)]) = [(j, x, t_j(x)t_{ji}(x)k)] = [(i, x, t'_{ij}(x)t_j(x)t_{ji}(x)k)]$$

and

$$[(i, x, t'_{ij}(x)t_j(x)t_{ji}(x)k)] = [(i, x, t_i(x)t_{ij}(x)t_{ji}(x)k)] = [(i, x, t_i(x)k)].$$

Clearly, f is continuous and there exists $f' : \tilde{X}_2 \to \tilde{X}_1$ such that $f'([(i, x, k)]) := [(i, x, t_i^{-1}(x)k)]$, which implies, $f \circ f' = id$ and $f' \circ f = id$. Hence, f is a homeomorphism.
Now, from $\tilde{X}_1 \cong \tilde{X}_2$, we claim that $t_i(x) = 1$ would work to show that E_1 and E_2 are isomorphic according to Proposition 3.22. This completes the correspondence between line bundles and double covers up to isomorphism. □

3.4 Examples

Example 3.49 (Trivial structures)**.** *Let us first see that the trivial structure are in correspondence with each other by the above mentioned correspondences. So, consider a trivial double cover* $(\tilde{X}, p)$ *of a topological space* X *which is path-connected. This will give us* $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong p_*(\pi_1(X)) \cong \pi_1(X)$*, which is the trivial subgroup of the fundamental group of space* X*, which is the required corresponding subgroup.*
Now, in order to define a trivial line bundle in terms of transition functions, first consider X *to be a base space, and consider any open covering* $\{U_\alpha\}_{\alpha \in I}$ *of* X *and define all the transition functions to be the constant 1 function. As per our construction above, the corresponding double cover looks like* $\tilde{X} = \{(i, x, k) | i \in I, x \in U_i, k = 1 \text{ or } -1\}$ *as a subspace of* $E = \{(i, x, v) | i \in I, x \in U_i, v \in \mathbb{R}\}$*. Note that there is no extra relation used while defining* E *since, all* t_{ij}*'s are the constant 1 function. So, in fact,* $\tilde{X} \cong X \sqcup X$ *and the covering map is* $p((x, k)) = x$ *for all* $x \in X, k = 1, -1$*, which is the same covering map, as that of the trivial double cover. Hence, even in this case the trivial structures correspond to each other.*

Example 3.50 ($X = \mathbb{S}^1$)**.** *For the second example, consider* $X = \mathbb{S}^1$*. We know that the fundamental group of* $\mathbb{S}^1$ *is* $\mathbb{Z}$*. Moreover, all the subgroups of* $\mathbb{Z}$ *are of the form* $n\mathbb{Z}$ *and the index of* $n\mathbb{Z}$ *in* $\mathbb{Z}$ *is* n*. Hence, there is only one subgroup (namely* $2\mathbb{Z}$*) of* $\mathbb{Z}$ *which is of index 2. Hence, by the correspondence developed above, we can say that there is unique non-trivial double cover (up to isomorphism) of the space* $\mathbb{S}^1$*, and also, there is unique non-trivial line bundle (up to isomorphism) of* $\mathbb{S}^1$*. We know that the squaring map from* $\mathbb{S}^1$ *to* $\mathbb{S}^1$ *is a non-trivial double cover, and the infinite Möbius strip is a non-trivial line bundle of* $\mathbb{S}^1$*. To verify this, consider the non-trivial double cover with squaring map as the covering map. Every loop in the domain runs with twice its speed and becomes concatenation of the same loop with itself, which gives us* $p_*(\pi_1(\mathbb{S}^1)) = 2\mathbb{Z}$*. This takes care of the correspondence between double covers and index two subgroups of the fundamental group. For the other correspondence, refer figure 7.*
The two figures on the leftmost side shows the squaring map from $\mathbb{S}^1$ *to* $\mathbb{S}^1$ *and let* $U_1 = \mathbb{S}^1 - \{-1\}, U_2 = \mathbb{S}^1 - \{i\}$*. So, we get* V_{11}, V_{12} *which are denoted in the figure at the center with single and double arrow respectively. Similarly,* V_{21}, V_{22} *are shown the rightmost figure with single and double arrows respectively.*

For this particular example, $\mathbb{S}^1$ *was divided into four parts a,b,c,d as shown in the figure and their corresponding inverses are also marked in it. According to our construction described earlier, the points in the parts a,b,c will get mapped to 1 under the transition function* (t_{12})*, and the points in part d, will get mapped to (-1). This is nothing but the Möbius band. Note that the continuity of the transition function is not affected since,* $\{-1, i\}$ *are not in its domain.*

For the other way round, given an infinite Möbius band $E = [0, 1] \times \mathbb{R} / \sim$*, such that* $(x, y) \sim (x, -y)$ *for all* $x \in [0, 1], y \in \mathbb{R}$ *considering the boundary of a Möbius band, we get* $\tilde{X} = [0, 1] \times \{-1, 1\} / \sim$ *such that* $(x, 1) \sim (x, -1)$ *for all* $x \in [0, 1]$ *and* $\mathbb{S}^1 \cong \{(x, 0) | x \in [0, 1]\} \subseteq E$ *we get a natural double covering from* $\tilde{X}$ *to* $\mathbb{S}^1$ *which is non-trivial.*

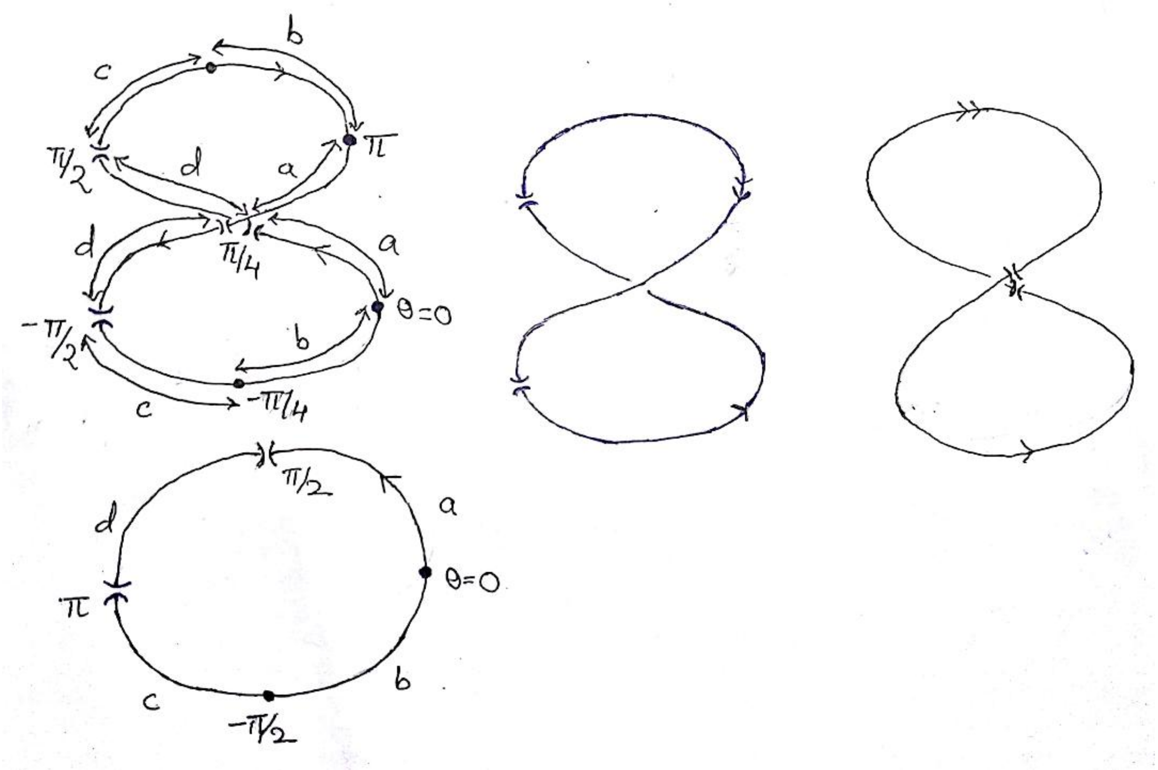


Figure 7: Non-trivial double cover of $\mathbb{S}^1$.

4 Vector bundles: Swan's theorem

In the mathematical fields of topology and K-theory, the Swan's theorem, also called Serre-Swan theorem, relates the geometric notion of vector bundles to the algebraic concept of projective modules and gives rise to a common intuition throughout mathematics: "projective modules over commutative rings are like vector bundles on compact spaces". The original theorem, as stated by Jean-Pierre Serre in 1955, is more algebraic in nature. The complementary variant stated by Richard Swan in 1962 is more analytic, and concerns vector bundles. We will be studying a proof of this theorem in this chapter. We will be referring [3, 10] for the proof of Swan's theorem.

4.1 Sections of a vector bundle

Definition 4.1 (Section). *Let (E, π) be a vector bundle over a topological space X. A section s of the vector bundle is a continuous map $s : X \to E$ such that $\pi \circ s = Id_X$ (that is, $s(x) \in E_x$, for all $x \in X$).*

Definition 4.2 (R-module). *Let R be a ring. A is called an R-module if A is an abelian group and there exists a map $R \times A \to A$ such that (i) $r(a + b) = ra + rb$, (ii) $(r + s)a = ra + rs$, (iii) $(rs)a = r(sa)$, (iv) if R has unity, then $1a = a$, for all $r, s \in R, a, b \in A$.*

Example 4.3. *Sections of a vector bundle (E, π) over a topological space X forms a $C(X)$-module since, we can define $(fs)(x) := f(x)s(x)$ for each $f \in C(X), s \in \Gamma(E)$ which satisfies the required properties.*

Definition 4.4 (R-linear map). *Let A, B be two R-modules. A map $\phi : A \to B$ is called R-linear map if $\phi(ra + b) = r\phi(a) + \phi(b)$ for all $r \in R, a, b \in A$.*
If ϕ is a bijection as well as R-linear, then it is called an isomorphism of R-modules.

Definition 4.5 (Direct sum). *Let $\{A_i\}_{i\in I}$ be a collection of R-modules. A direct sum of $\{A_i\} := \{(a_i)_{i\in I} | a_i \in A_i$ for all $i \in I, a_i = 0$ for all but finitely many $i \in I\}$. It is denoted by $\bigoplus_{i\in I} A_i$. Moreover, if I is finite, say, $|I| = n$ then $\bigoplus_{i\in I} A$ is also denoted as A^n.*

Proposition 4.6. *Let $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over a topological space X. Then $\Gamma(E_1 \oplus E_2) \cong \Gamma(E_1) \oplus \Gamma(E_2)$ as $C(X)$-modules.*

Proof. Define two maps of vector bundles $i_1 : E_1 \to E_1 \oplus E_2$ as $a \mapsto (a, S_2(\pi_1(a)))$, and $i_2 : E_2 \to E_1 \oplus E_2$ as $b \mapsto (S_1(\pi_2(b)), b)$, where, $S_i : X \to E_i$ such that $x \mapsto (x, 0)$ are the zero sections, for $i = 1, 2$. Note that $\pi_1(a) = \pi_2(S_2(\pi_1(a)))$ since, $\pi_2 \circ S_2 = Id_X$, hence, i_1 is well-defined. Similarly $\pi_2(b) = \pi_1(S_1(\pi_2(b)))$ since, $\pi_1 \circ S_1 = Id_X$, hence i_2 is also well-defined. In other words, $i_1 = Id \times 0$, and $i_2 = 0 \times Id$, hence they indeed are maps of vector bundles.
Now, for $S_1 \in \Gamma(E_1)$, define, $\tilde{S}_1 := i_1 \circ S_1 \in \Gamma(E_1 \oplus E_2)$ and for $S_2 \in \Gamma(E_2)$, define, $\tilde{S}_2 := i_2 \circ S_2 \in \Gamma(E_1 \oplus E_2)$. Now, for any $S \in \Gamma(E_1 \oplus E_2)$, we can define

$\tilde{S}_1 := i_1 \circ (pr_1 \circ S), \tilde{S}_2 := i_2 \circ (pr_2 \circ S)$ which implies $S = \tilde{S}_1 + \tilde{S}_2$. Now, we claim that $\{i_1 \circ S_1\} \cap \{i_2 \circ S_2\} = 0$. If $i_1 \circ S_1 = i_2 \circ S_2$ for some $S_1 \in \Gamma(E_1), S_2 \in \Gamma(E_2)$, then applying pr_1 on both the sides give us $pr_1 \circ i_1 \circ S_1 = pr_1 \circ i_2 \circ S_2$, which implies S_1 is equal to the zero section, hence, $i_1 \circ S_1 = 0$, which proves the claim. Clearly, $\{i_1 \circ S_1\}_{S_1 \in \Gamma(E_1)} \cong \Gamma(E_1)$, and $\{i_2 \circ S_2\}_{S_2 \in \Gamma(E_2)} \cong \Gamma(E_2)$, hence, $\Gamma(E_1 \oplus E_2) \cong \Gamma(E_1) \oplus \Gamma(E_2)$. □

Remark 4.7. *An element S in the set of sections $\Gamma(E_1 \otimes E_2)$ looks like $S(x) = S_1(x) \otimes S_2(x)$ for some $S_1 \in \Gamma(E_1), S_2 \in \Gamma(E_2)$, for all $x \in X$.*

Proposition 4.8. *Let $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over a topological space X. Then, $\Gamma(E_1) \otimes_{C(X)} \Gamma(E_2) \cong \Gamma(E_1 \otimes E_2)$ as $C(X)$-modules.*

Proof. Let us define a map $\Phi : \Gamma(E_1) \otimes_{C(X)} \Gamma(E_2) \to \Gamma(E_1 \otimes E_2)$ as

$$S_1 \otimes_{C(X)} S_2 \mapsto (\Phi(S_1 \otimes S_2)(x) := S_1(x) \otimes S_2(x) \text{ for all } x \in X)$$

It is easy to check that Φ is well-defined over equivalence classes under bilinearity since there is tensor product structure in the codomain as well. Similarly, we can see that Φ is a $C(X)$-linear map. Now, to prove that Φ is injective, let $\Phi([S_1 \otimes S_2]) = 0$, that is, $\Phi(S_1 \otimes S_2)(x) = S_1(x) \otimes S_2(x) = 0$ for all $x \in X$. This implies, for each $x \in X$, either $S_1(x) = 0$ or $S_2(x) = 0$, that is, $[(S_1(x), S_2(x))] = 0$ for all $x \in X$, where $[\,\cdot\,]$ is due to bilinearity at a fixed point $x \in X$, that is, over $\mathbb{R}$. This implies $[(S_1, S_2)] = 0$. Surjectivity comes immediately from remark 4.7. Hence we get a $C(X)$-linear isomorphism between $\Gamma(E_1) \otimes_{C(X)} \Gamma(E_2)$ and $\Gamma(E_1 \otimes E_2)$. □

Proposition 4.9. *E is a vector bundle of rank n over a topological space X if and only if for each $x \in X$, there exists $U \subseteq X$, an open set containing x and sections $\{s_1, s_2, \cdots, s_n\}$ on U such that $\{s_1(y), s_2(y), \cdots, s_n(y)\}$ is a basis for E_y for each $y \in U$.*

Proof. ($\Rightarrow$) Let (U, ϕ) be a chart such that $x \in U$. Then $\phi : E|_U \to U \times \mathbb{R}^n$ is a homeomorphism. Let $\{e_1, e_2, \cdots, e_n\}$ be the standard basis for $\mathbb{R}$. Define, $s_i(y) := \phi^{-1}(y, e_i)$ for each $i = 1, 2, \cdots, n$. Fixing y, we get s_i's to be linear isomorphisms. The linear independence of $\{s_1, s_2, \cdots, s_n\}$ follows from linear independence of $\{e_1, e_2, \cdots, e_n\}$, and since $s_1, s_2, \cdots, s_n$ are n in number, so it is indeed a basis for E_y.
($\Leftarrow$) Define a homeomorphism $\phi : U \times \mathbb{R}^n \to E|_U$ as follows:
$\phi(u, v_1, v_2, \cdots, v_n) := \sum_{i=1}^{n} v_i \cdot s_i(u)$. Since this is a linear isomorphism after fixing a $u \in U$, we get a required trivialization. □

Remark 4.10. *A vector bundle E is trivial if and only we can find such sections defined globally.*

4.2 Some properties of Hausdorff spaces

Lemma 4.11. *Let T be a Hausdorff topological space, $x \in T$, $Y \subseteq T$ compact such that $x \notin Y$. Then there exists $U, V \subseteq T$ open such that $x \in U$, $Y \subseteq V$, and $U \cap V = \varnothing$.*

Proof. Since T is Hausdorff, for every $y \in Y$, there exists a neighbourhood U_y of y and V_y of x such that $U_y \cap V_y = \varnothing$. So, $\{U_y\}_{y \in Y}$ is an open cover of Y. Hence, compactness of Y implies that there exists a finite subcover $\{U_{y_i}\}_{i=1}^n$. Define $U := \bigcup_{i=1}^{n} U_{y_i}$, $V := \bigcap_{i=1}^{n} V_{y_i}$. Clearly, $x \in V, Y \subseteq U$; U, V are open. Now, if $z \in U$, then $z \in U_{y_j}$ for some j, then $V \subseteq V_{y_j}$ and $V_{y_j} \cap U_{y_j} = \varnothing$, which implies $z \notin V$. Hence, $U \cap V = \varnothing$. □

Lemma 4.12. *If T is a Hausdorff topological space, and let Y be a compact set, then Y is closed.*

Proof. It is equivalent to prove that Y^C is an open set. If we consider a point $x \in Y^C$, then by previous lemma we can get an open set containing x, which is contained in Y^C. Since x was an arbitrary point in X, we are done. □

Definition 4.13. *(i) A topological space X is said to be T_1 if for any given two distinct points x, y we can find neighbourhoods U_x, U_y around x, y respectively, such that $x \notin U_y$ and $y \notin U_x$. In other words, any singleton sets $\{x\}$ is a closed set.*
(ii) A topological space is said to be normal if it is T_1 and for any given disjoint pair of closed subsets $A, B \subseteq X$ there exists disjoint open sets $C, D \subseteq X$ such that $A \subseteq C, B \subseteq D$.

Lemma 4.14. *If T is a compact, Hausdorff topological space, then T is normal.*

Proof. Let $X, Y \subseteq T$ be closed and disjoint. Then, by lemma 4.11, for any $y \in Y$, there exists U_y containing y and O_y such that $X \subseteq O_y$, $U_y \cap O_y = \varnothing$. So, $\{U_y\}_{y \in Y}$ is an open cover of Y. Hence, compactness of Y implies that there exists a finite subcover $\{U_{y_i}\}_{i=1}^m$. Now, define $O_1 := \bigcup_{i=1}^{m} U_{y_i}$, $O_2 := \bigcap_{i=1}^{m} O_{y_i}$ which implies O_1, O_2 are disjoint open sets such that $X \subseteq O_2, Y \subseteq O_1$. Hence, T is normal. □

4.3 Set of sections as a $C(X)$-module

Definition 4.15. *An R-module A is called free if $A \cong \bigoplus_{i \in I} R$ for some I (possibly uncountable).*
A is called finite free if $A \cong \bigoplus_{i \in J} R$ for some finite set J.
A is called projective if there exists an R-module B such that $A \oplus B$ is free.
A is called finite rank projective if there exists an R-module B such that $A \oplus B$ is finite free.

Definition 4.16 (Finitely generated R-module)**.** *An R-module A is called finitely generated if there exists a surjective map $\phi : R^n \to A$ for some $n \in \mathbb{N}$.*

From the definitions above and remark 4.12, we can say that a vector bundle (E, π) over a topological space X is trivial if and only if $\Gamma(E)$ is free and finitely generated $C(X)$-module.

Definition 4.17 (Short exact sequence)**.** *Let A, B, C be groups, $q : A \to B, r : B \to C$ be homomorphisms, then*

$$0 \longrightarrow A \xrightarrow{q} B \xrightarrow{r} C \longrightarrow 0$$

is called a short exact sequence if q is an injection, r is a surjection, and $im(q) = ker(r)$.

Lemma 4.18 (Splitting lemma)**.** *[4] Let A, B, C be abelian groups and*

$$0 \longrightarrow A \xrightarrow{q} B \xrightarrow{r} C \longrightarrow 0$$

be a short exact sequence, then the following are equivalent:
(i) Left split: There exists a homomorphism $t : B \to A$ such that $t \circ q = Id_A$.
(ii) Right split: There exists a homomorphism $u : C \to B$ such that $r \circ u = Id_C$.
(iii) Direct sum: There exists an isomorphism $h : B \to A \oplus C$ such that $h \circ q : A \to A \oplus C$ is the natural injection of A, and $r \circ h^{-1} : A \oplus C \to C$ is the natural projection onto C.

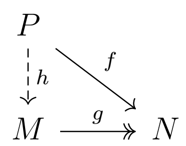
Proof. (iii) ⇒ (i) Define a homomorphism $t : B \xrightarrow{h} A \oplus C \to A$, i.e. $t = pr_1 \circ h$ which implies $t \circ q = Id_A$.

(iii) ⇒ (ii) Define a homomorphism $u : C \hookrightarrow A \oplus C \to B$, that is, $t = h^{-1} \circ i$ which implies $r \circ u = Id_C$.

(i) ⇒ (iii) Note that, any $b \in B$ can be written as $im(q) + ker(t)$, since, $b = q(t(b)) + (b - q(t(b)))$ (see that $(b - q(t(b))) \in ker(t)$ since, $t(b - q(t(b))) = t(b) - t(q(t(b))) = t(b) - t(b)$ as $t \circ q = Id_A$). Now we claim that $im(q) \cap ker(t) = \{0\}$. Let $b \in im(q)$, then $b = q(a)$ for some $a \in A$, now $b \in ker(t)$ implies $t(b) = 0$, which implies $t(q(a)) = 0$, and since $t \circ q = Id_A$, $t(q(a)) = a = 0$, hence, $b = q(a) = 0$, as q is a homomorphism. Hence, $B \cong im(q) \oplus ker(t)$, so for each $b \in B$ there exists unique $b' \in im(q)$ and $k \in ker(t)$ such that $b = b' + k$ and using injectivity of q, there exists unique $a \in A$ such that $b = q(a) + k$. Now, since r is onto, for any $c \in C$ there exists $b = q(a) + k$ such that $r(b) = c$, which implies $r(q(a) + k) = c$, which implies $r(k) = c$ since $(r(q(a)) = 0$, as $im(q) = ker(r)$ (due to exactness). So, for any $c \in C$ there exists $k \in ker(t)$ such that $r(k) = c$. Hence, $r|_{ker(t)} : ker(t) \to C$ is onto. Now, if, for some $k \in ker(t)$, $r(k) = 0$, that is, $k \in ker(r) = im(q)$, which implies $k \in im(q) \cap ker(t)$, which implies $k = \{0\}$. Hence, $r|_{ker(t)} : ker(t) \to C$ is injective and hence an isomorphism. So, $ker(t) \cong C$. Now, note that $q : A \to im(q)$ is injective, since q is so. Also, this map is surjective. Hence, $A \cong im(q)$. Hence, $B \cong A \oplus C$.

(ii) $\Rightarrow$ (iii) We will use similar argument here as well. Any $b \in B$ can be written as $ker(r) + im(u)$ since, $b = (b - u(r(b))) + u(r(b))$. Now, let $a \in im(u) \cap ker(r)$ then, $a = u(c)$ for some $c \in C$ and $r(a) = 0$, which implies, $r(u(c)) = 0$, which implies $c = 0$ since, $r \circ u = Id_C$. Hence, $a = u(c) = 0$ as u is a homomorphism. This proves that $B \cong ker(r) \oplus im(u)$. In previous part, we saw that $im(q) \cong A$ and $im(q) = ker(r)$ (due to exactness) implies $ker(r) \cong A$. Now, since, $r \circ u (= Id_C)$ is a bijection, u is an injection. Hence, $u : C \to im(u)$ is an isomorphism, that is, $im(u) \cong C$, which proves that $B \cong A \oplus C$. □

Proposition 4.19. *[4] Let R be a ring with unity and P be an R-module. Then the following are equivalent.*
(i) P is a projective module.
(ii) For any homomorphism $f : P \to N$, and a surjective homomorphism $g : M \to N$, there exists a homomorphism $h : P \to M$ such that the following diagram commutes.



(iii) Every short exact sequence of the following form splits:

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

Proof. (i) $\Rightarrow$ (ii) Let Q be an R-module such that $P \oplus Q$ is a free module. Let $B = \{b_i\}_{i \in I}$ be a basis of $P \oplus Q$. We can say that the basis exists due to the assumption of the existence of unity. Since g is a surjective map, for each $i \in I$, there exists $m_i \in M$ such that $f(pr_1(b_i)) = g(m_i)$. So, we can define a map $H : P \oplus Q \to M$ as $H\Big(\sum_{i=1}^{n} r_i b_i\Big) = \sum_{i=1}^{n} r_i m_i$ where $r_i \in R$ for all $i \in I$. Note that this map is indeed well defined since we are defining H on the basis $\{b_i\}_{i \in I}$ and extending it linearly to whole of $P \oplus Q$. Now, define $h : P \to M$ such that $h := H|_P$. Then we get $g \circ h = f$, hence we are done.
(ii) $\Rightarrow$ (iii) Let

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence, which means $g : M \to P$ is a surjective map. So, we can consider the following diagram:

$$\begin{array}{ccc} P & & \\ & \searrow^{Id} & \\ M & \xrightarrow{g} & P \end{array}$$

That is, we have considered $N = P$, and $f = Id$ in (ii). Hence, by (ii) there exists a homomorphism $h : P \to M$ such that $g \circ h = Id_P$ which implies there exists a right split. Hence, the short exact sequence splits.
(iii) $\Rightarrow$ (i) We have the canonical surjection of R-modules as follows:
$f : \bigoplus_{p \in P} R \to P$, where an element $((r_p)_{p \in P})$ gets mapped to $\sum_{p \in P} r_p p$. Note that the sum makes sense because r_p's are zero for all but finitely many p's. Clearly, it is an R-module homomorphism as well as a surjection. This gives us a short exact sequence as follows:

$$0 \longrightarrow ker(f) \hookrightarrow \bigoplus_{p \in P} R \xrightarrow{f} P \longrightarrow 0$$

Now, by (iii) this short exact sequence splits, hence we can write $ker(f) \oplus P \cong \bigoplus_{p \in P} R$. Hence, there exists an R-module, Q, namely, $ker(f)$ such that $P \oplus Q$ is free. This implies P is a projective module. $\square$

Lemma 4.20. *An R-module A is finite rank projective module if and only if A is finitely generated projective module.*

Proof. ($\Rightarrow$) Since A is finite rank projective module, there exists an isomorphism ϕ: $R^n \to A \oplus B$ for some $n \in \mathbb{N}$, for some R-module B. Moreover, A is finitely generated since $pr_1 \circ \phi : R^n \to A$ is a surjection. Also, it is projective by definition.
($\Leftarrow$) Since A is finitely generated, there exists a surjection $\phi : R^n \to A$ for some $n \in \mathbb{N}$. Then, using proposition 4.19 we can say that, the short exact sequence

$$0 \longrightarrow ker(\phi) \hookrightarrow R^n \xrightarrow{\phi} A \longrightarrow 0$$

spits, since A is a projective module. Hence, we get $ker(\phi) \oplus A \cong R^n$ which concludes that A is a finite rank projective module. $\square$

Proposition 4.21. *A finitely generated R-module A is finite rank projective if and only if there exists an idempotent $P \in M_n(R)$ (that is, $P^2 = P$) such that $A \cong P(R^n)$ for some $n \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Since A is finitely generated, there exists a surjection $\phi : R^n \to A$ and since, A is finite rank projective R-module, there exists an R-module B and $m \in \mathbb{N}$ such that $A \oplus B \cong R^m$ (let ψ: $A \oplus B \to R^m$ denote the isomorphism). Note that, the map $pr_1 \circ \psi^{-1} : R^m \to A$ is also a surjection. So, we can consider $\phi = pr_1 \circ \psi$, and $n = m$. Then we have a sequence of abelian groups:

$$0 \longrightarrow ker(\phi) \hookrightarrow R^n \longrightarrow A \longrightarrow 0$$

where, the map from $R^n \to A$ is the map ϕ. Since, ϕ is surjection. Hence, the sequence mentioned above is a short exact sequence of abelian groups and there exists a right split, since we can define a map $t : A \to R^n$ as follows: $A \hookrightarrow A \oplus B \to R^m \xrightarrow{Id} R^n$ which implies $\phi \circ t = Id_A$. Now, by splitting lemma, we have $A \oplus ker(\phi) \cong R^n$ (let $\Phi : A \oplus ker(\phi) \to R^n$ denote the isomorphism). Then similar to previous case, we can define $P = \Phi \circ pr_1 \circ \Phi^{-1} \in M_n(R)$ which follows $P^2 = P$. Now, in order to show $P(R^n) \cong A$, let $c \in R^n$, then P acts on c as follows: $c \mapsto (a, b) \mapsto a \mapsto \Phi(a)$. So, $P(R^n) \cong \{\Phi(a) | a \in A\} \cong A$ since Φ is an isomorphism.
($\Leftarrow$) Claim: $A \oplus (1-P)(R^n) \cong R^n$. Let $c \in R^n$, $c = c + P(c) - P(c) = P(c) + (1-P)(c)$, where $P(c) \in A$ since $A \cong P(R^n)$. Now, to see that $A \cap (1-P)(R^n)$, let $a \in A \cap (1-P)(R^n)$, which implies $a = b - P(b) = P(c)$ for some $b, c \in R^n$. Applying P on both the sides, we get $P^2(c) = P(b - P(b))$ implies $P(c) = P(b) - P^2(b) = P(b) - P(b) = 0$. Hence, $a = P(c) = 0$. This proves the claim and the claim directly implies that A is finite rank projective module and $pr_1 : R^n \cong A \oplus (1-P)(R^n) \to A$ gives us a surjection required to prove the finitely generatedness. □

4.4 Swan's theorem

Theorem 4.22 (Swan's theorem). *[22] Let X be a compact, Hausdorff topological space. Then there exists a one-to-one correspondence between vector bundles over X, and finitely generated projective $C(X)$-modules. The correspondence is given by $E \mapsto \Gamma(E)$.*

Proof. We will use a series of lemmas to prove this theorem as follows.

Lemma 4.23. *Let (E, π) be a vector bundle over a compact, Hausdorff topological space X. Then $\Gamma(E)$ is a finitely generated $C(X)$-module.*

Proof. Let $\{U_i\}_{i \in I}$ be a finite cover of trivializing neighbourhoods, and let $\{\rho_i\}_{i \in I}$ be a partition of unity subordinate to $\{U_i\}_{i \in I}$. (This exists due to proposition 3.35). By definitions and remark 4.12 we know that $\Gamma(E|_{U_i})$ is free and finitely genareated. If $s \in \Gamma(E|_{U_i})$ then extend s to all E by

$$\tilde{s}(x) := \begin{cases} \rho_i(x) \cdot s(x) & \text{if } x \in U_i \\ 0, & \text{otherwise} \end{cases}$$

Since, U_i's are generated by finitely many sections (by proposition 4.9) and there are finitely many U_i's, hence, $\Gamma(E)$ is finitely generated. In other words, let $t \in \Gamma(E)$, then we can write $t(x) = \sum_{i \in I} \rho_i(x) \cdot t(x)$. Now, each $\rho_i(x) \cdot t(x) \in \Gamma(E|_{U_i})$ and $\Gamma(E|_{U_i})$ is finitely generated due to the definitions and remark 4.12. Hence, we are done. □

Lemma 4.24. *Let (E, π) be a vector bundle over a compact topological space X. Then there exists $(E^\perp, \pi^\perp)$, a vector bundle over X such that $E \oplus E^\perp \cong X \times \mathbb{R}^m$ for some $m \in \mathbb{N}$.*

Proof. Let $\{U_i\}_{i=1}^n$ be a finite open cover of trivializing neighbourhoods with trivializations $\{\phi_i\}_{i=1}^n$ and $\{\rho_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_i\}_{i=1}^n$. If rank of (E, π) is k, then define $\Phi : E \to X \times \mathbb{R}^{nk}$, a map of vector bundles such that

$$(e) \mapsto (\pi(e), \rho_1^{1/2}(\pi(e))\phi_1(e), \cdots, \rho_n^{1/2}(\pi(e))\phi_n(e))$$

Now, let us prove that Φ is an injective map. So, let $e_1, e_2 \in E$ be two elements such that $\Phi(e_1) = \Phi(e_2)$. So, equating the first co-ordinate in $X \times \mathbb{R}^{nk}$ we get $\pi(e_1) = \pi(e_2)$, meaning e_1 and e_2 lie on a same fiber. Now, since the partition of unity $\{\rho_i\}_{i=1}^n$ are non-negative functions which add up to one for each point, we get a function ρ_j such that ρ_j is positive at the point $\pi(e)$. Moreover, we have $\rho_i^{1/2}(\pi(e_1))\phi_i(e_1) = \rho_i^{1/2}(\pi(e_2))\phi_i(e_2)$ for all $i \in \{1, 2, \cdots, n\}$. In particular,

$$\rho_j^{1/2}(\pi(e_1))\phi_j(e_1) = \rho_j^{1/2}(\pi(e_2))\phi_j(e_2)$$

and we can cancel $\rho_j^{1/2}(\pi(e_k))$, $k = 1, 2$ from both the sides since they are equal and non-zero, which leaves us the fact that $\phi_j(e_1) = \phi_j(e_2)$ which in turn implies $e_1 = e_2$ since, ϕ_i's are injections since, by definition, they are linear isomorphisms over fixed E_x. Hence, E embeds in $X \times \mathbb{R}^{nk}$. Now, we use the Euclidean structure over $X \times \mathbb{R}^{nk}$ to define an orthogonal projection: $P_x : \{x\} \times \mathbb{R}^{nk} \to E_x$. Now, using continuity of the Euclidean structure over X, we can have P (over all of X) to be continuous, and we can define $E^\perp := (1 - P)(X \times \mathbb{R}^{nk})$ which implies $E \oplus E^\perp \cong X \times \mathbb{R}^{nk}$. □

Proposition 4.25. *Let (E, π) be a vector bundle over a compact, Hausdorff topological space X. Then $\Gamma(E)$ is finitely generated and projective $C(X)$-module.*

Proof. The module $\Gamma(E)$ is finitely generated due to lemma 4.23. Now, we will use lemma 4.20, so it is enough to prove the finite rank projectiveness. In order to prove the finite rank projectiveness, we use lemma 4.24 and proposition 4.6, which implies $\Gamma(E) \oplus \Gamma(E^\perp) \cong \Gamma(X \times \mathbb{R}^{nk})$, and remark 4.12 with the definitions imply $\Gamma(X \times \mathbb{R}^{nk}) \cong (C(X))^{nk}$. Hence we are done. □

Now, let us prove the other side of the correspondence, that is, starting from finitely generated projective $C(X)$-module A, we have to construct a vector bundle (E, π) such that $\Gamma(E) \cong A$.
Using proposition 4.21, we have a correspondence between finitely generated projective modules over $C(X)$ with idempotents $P \in M_n(C(X))$. We can look at as the set

$$\{f : X \to M_n(\mathbb{R}) | \, f \text{ is continuous}, f(x) \text{ is an idempotent matrix in } M_n(\mathbb{R}), \forall x \in X\}$$

So, using lemma 4.20 we can say that, in order to prove the Swan's theorem, it is sufficient to prove the following lemma:

Lemma 4.26. *If $P : X \to M_n(\mathbb{R})$ is an idempotent valued continuous function, then $Im(P) := \{(x, v) \in X \times \mathbb{R}^n | v \in Range(P(x))\}$ is a vector bundle over X equipped with the subspace topology (as, $Im(P) \subseteq X \times \mathbb{R}^n$), and $\pi : Im(P) \to X$ is pr_1. Moreover, $\Gamma(Im(P)) = P((C(X))^n)$.*

Proof. If we consider $\pi : Im(P) \to X$ as pr_1, then $\pi^{-1}(x) = Range(P(x))$ which is a vector space. So, fiberwise vector space structure is clear. Now, we have to prove local triviality. Let $x_0 \in X$, and let $Range(P(x_0))$ be a k-dimensional subspace of $\mathbb{R}^n$. Let $\{v_1, v_2, \cdots, v_k\}$ be its basis; extend it to $\{v_1, v_2, \cdots, v_n\}$ as basis of $\mathbb{R}^n$. Now, the following matrix valued function $\tilde{P} : X \to M_n(\mathbb{R})$

$$\tilde{P}(x) := [P(x)v_1|P(x)v_2|\cdots|P(x)v_k|v_{k+1}|\cdots|v_n]$$

is invertible at $x = x_0$, since the columns are linearly independent as $\{v_1, \cdots, v_n\}$ is a basis for $\mathbb{R}^n$ and $P(x_0)v_1, P(x_0)v_2, \cdots, P(x_0)v_k$ are non-zero vectors as they lie in $Range(P(x_0))$. Now, note that $\tilde{P}$ is a continuous function, and since $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$ (as the determinant map $det : M_n(\mathbb{R}) \to \mathbb{R}$ is a polynomial function, and hence it is continuous and $GL_n(\mathbb{R})$ is pre-image of an open set, namely $\mathbb{R} - \{0\}$), there exists an open set $U \subseteq X$ containing x_0 such that $\tilde{P}(x)$ is invertible for all $x \in U$. In particular, for all $x \in U$ all $P(x)v_1, \cdots, P(x)v_k$ are linearly independent, so, setting $S_i(x) := P(x)v_i$, $x \in U, i = 1, 2, \cdots, k$ gives sections of $E|_U$ which are basis for each fiber E_x. Hence, using proposition 4.9, we get a vector bundle. Now, let $s \in \Gamma(Im(P))$ which is equivalent to $s(x) \in Range(P(x))$ which is equivalent to $s(x) = P(x)v(x)$ for some $v(x) \in \mathbb{R}^n$ which is equivalent to $s \in P((C(X))^n)$. Hence, $\Gamma(Im(P)) = P((C(X))^n)$. □

This proves Swan's theorem. □

Now, one naturally expects that this corresponcence will hold even for the isomorphism classes of vector bundles and isomorphism classes of finitely generated projective modules. So, the following theorem proves this fact when the base space is normal. Note that this theorem works in our case as well, since a compact Hausdorff space is normal.

Theorem 4.27. *Let X be a normal space, and $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over the space X of rank k. Then (E_1, π_1) is isomorphic to (E_2, π_2) if and only if $\Gamma(E_1)$ is isomorphic to $\Gamma(E_2)$.*

Proof. ($\Rightarrow$) Let $f : E_1 \to E_2$ be a homeomorphism. Then construct $\psi : \Gamma(E_1) \to \Gamma(E_2)$ as $S \mapsto f \circ S$. This map is linear since f is linear after fixing a point $x \in X$. Also, we can construct $\psi^{-1} : \Gamma(E_2) \to \Gamma(E_1)$ as $S \mapsto f^{-1} \circ S$. Also, it is evident that $\psi \circ \psi^{-1} = Id$, and $\psi^{-1} \circ \psi = Id$. Hence, ψ can act as a required isomorphism. Note that this part of the proof is valid for any topological space X.
($\Leftarrow$) Let $\phi : \Gamma(E_1) \to \Gamma(E_2)$ be an isomorphism. We claim that there exists a map $\psi : E_1 \to E_2$ as follows: $(p, v) \mapsto (\phi(S))(p)$ where $S \in \Gamma(E_1)$ such that $S(p) = \phi_p^{-1}(p, v)$, where ϕ_p is a trivialization on a trivializing neighbourhood around p. Let us check the linearity of ψ after fixing a point $p \in X$. Consider $(p, v_1 + v_2)$, if $S_1, S_2 \in \Gamma(E_1)$ such that $S_1(p) = (p, v_1)$, and $S_2(p) = (p, v_2)$ then by construction, $S_1 + S_2 \in \Gamma(E_1)$ such that $(S_1 + S_2)(p) = (p, v_1 + v_2)$. So,

$$\psi(p, v_1 + v_2) = (\phi(S_1 + S_2))(p) = (\phi(S_1) + \phi(S_2))(p)$$

since ϕ is linear and $(\phi(S_1) + \phi(S_2))(p) = \phi(S_1)(p) + \phi(S_2)(p)$ which comes from the definition of addition in $\Gamma(E_2)$. This gives us $\psi(p, v_1) + \psi(p, v_2)$. Note that the same proof works for $\psi(p, \alpha \cdot v) = \alpha \cdot \psi(p, v)$ where $\alpha \in \mathbb{R}$. Now, let us prove that for each $(p, v) \in E_1$ there exists $S \in \Gamma(E_1)$ such that $S(p) = (p, v)$. Let (p, v) be fixed. Let U be a trivializing neighbourhood containing p, let $\phi : E|_U \to U \times \mathbb{R}^k$ be the corresponding trivialization. Since X is completely regular space since it is a normal space, there exists a continuous function $f : X \to [0, 1]$ such that $f(p) = 1, f(U^C) = \{0\}$. Define a section S as $S(x) := \phi_x^{-1}(x, f(x)v)$ where ϕ_x is a trivialization over a neighbourhood containing x. Note that the section is well-defined, that is, it does not get affected by the choice of ϕ_x's since they are defined in that manner in the definition of a vector bundle. Now, let us prove that ψ is well-defined. So, let $S_1, S_2 \in \Gamma(E_1)$ be two distinct sections such that $S_1(p) = S_2(p) = \phi_p^{-1}(p, v)$. We need to prove that $(\phi(S_1))(p) = (\phi(S_2))(p)$. Using linearity of ψ we can say that it is enough to show that, if $S(x) = \phi_x^{-1}(x, 0)$ then $(\phi(S))(x) = \psi_x^{-1}(x, 0)$ where, ψ_x is a trivialization in E_2 on a trivializing neighbourhood around x. In order to prove this, let us consider a map of vector spaces over $\mathbb{R}$ as $\Phi : \Gamma(E) \to E_x$ such that $S \mapsto S(x)$. We will show that $ker(\Phi) = \mathfrak{m}_x \cdot \Gamma(E)$. For then, if $S(x) = 0$, then $S = \sum_{i=1}^{n} f_i \cdot S_i'$ where, $f_i \in \mathfrak{m}_x, S_i' \in \Gamma(E)$, hence

$$(\phi(S))(x) = \Big(\phi\Big(\sum_{i=1}^{n} f_i \cdot S_i'\Big)\Big)(x) = \sum_{i=1}^{n} (f_i(x) \cdot (\phi(S_i'))(x)) = 0$$

because $f_i(x) = 0$, for all $i \in \{1, 2, \cdots, n\}$ as $f_i \in \mathfrak{m}_x$ for all $i \in \{1, 2, \cdots, n\}$. So, let $\sum_{i=1}^{n} f_i \cdot S_i \in \mathfrak{m}_x \cdot \Gamma(E)$ be an element. Applying Φ on it, we get

$$\Phi\Big(\sum_{i=1}^{n} f_i \cdot S_i\Big) = \Big(\sum_{i=1}^{n} f_i \cdot S_i\Big)(x) = \sum_{i=1}^{n} f_i(x) \cdot S_i(x) = 0$$

This implies $\sum_{i=1}^{n} f_i \cdot S_i \in ker(\Phi)$. For the reverse inclusion, let $S \in \Gamma(E)$ such that $S(x) = 0$. By proposition 4.9 we can write $S = \sum_{i=1}^{k} f_i \cdot S_i$ where, $f_i \in C(U), S_i \in \Gamma(E|_U)$. Note that it is an equality as sections over U. Now, note that, by lemma 3.32 there exists an open set V such that $V \subseteq \overline{V} \subseteq U$. So, since X is normal, using the Urysohn's lemma, there exists a continuous function $g : X \to [0, 1]$ such that $g(\overline{V}) = \{1\}, g(U^C) = \{0\}$. Multiplying f_i's and S_i's with g we get $\tilde{f}_i \in C(X)$ such that $\tilde{f}_i|_V \equiv f_i$, $\tilde{S}_i \in \Gamma(E)$ such that $\tilde{S}_i|_V \equiv S_i$. Now define a global section $S' := S - \sum_{i=1}^{k} \tilde{f}_i \cdot \tilde{S}_i$. Note that $S'|_V \equiv 0$. Since X is completely regular as it is normal, there exists a continuous function $a : X \to [0, 1]$ such that $a(x) = 0, a(V^C) = \{1\}$. This implies that $S' = a \cdot S'$ simply because S' is zero on V and a is one outside V.

So we can write $S = a \cdot S' + \sum_{i=1}^{k} \tilde{f}_i \tilde{S}_i \in \mathfrak{m}_x \cdot \Gamma(E)$ since $a \in \mathfrak{m}_x$, and $\tilde{f}_i \in \mathfrak{m}_x$ because S_i's are linearly independent so, $S(x) = 0$ implies $f_i(x) = 0$ for all $i \in \{1, 2, \cdots, k\}$, and so are $\tilde{f}_i(x) = 0$. This proves that ψ is well-defined. Now, we will simply define a function $\psi^{-1} : E_2 \to E_1$ as $(p, v) \mapsto (\phi^{-1}(S))(p)$ where $S \in \Gamma(E_2)$ such that $S(p) = (p, v)$. Now, $\psi^{-1} \circ \psi(p, v) = \psi^{-1}((\phi(S))(p))$. Let $t := \phi(S)$. Then $\psi^{-1}((\phi(S))(p)) = \psi^{-1}(p, v') = \phi^{-1}(S')(p)$ where $S' \in \Gamma(E_2)$ such that $S'(p) = (p, v')$ but the equation above implies that $S' = t = \phi(S)$ would work. So, $\phi^{-1}(S')(p) = \phi^{-1}(\phi(S))(p) = S(p) = (p, v)$. Hence, $\psi^{-1} \circ \psi = Id$. Similarly, $\psi \circ \psi^{-1}(p, v) = \psi((\phi^{-1})(S))(p)$, let $t := \phi^{-1}(S)$. Then $\psi(p, v') = (\phi(S'))(p)$ where $S' \in \Gamma(E_1)$ such that $S'(p) = (p, v')$ but the equation above implies that $S' = t = \phi^{-1}(S)$ would work. So, $(\phi(S))(p) = (\phi(\phi^{-1}(S)))(p) = S(p) = (p, v)$. This implies $\psi \circ \psi^{-1} = Id$. Now it remains to prove that ψ is continuous. Let $\pi_1 : E_1 \to X$ be the projection. Let $e \in E_1$ and $x = \pi_1(e) \in X$. According to proposition 4.9 there exists a neighbourhood U of x such that there exist sections $S_1, S_2, \cdots, S_k \in \Gamma(E_1)$ such that $S_1(y), S_2(y), \cdots, S_k(y)$ is a basis for E_y for each $y \in U$. So, let any point $e' \in \pi_1^{-1}(U)$ with projection $x' = \pi_1(e')$ can be written as $e' = \sum_{i=1}^{k} f_i S_i(x')$ and S_i's $\in C(U)$. But using the fact that X is completely regular as it is normal, we can extend S_i's by multiplying by a function which is zero outside U, and one at x' to get sections. Call these sections $\tilde{S}_i$. Now note that, if we consider $S = \sum_{i=1}^{k} f_i(x') \tilde{S}_i \in \Gamma(E_1)$, then $S(x') = e'$, so

$$\psi(e') = (\phi(S))(x') = \Big(\sum_{i=1}^{k} f_i(x') \phi(\tilde{S}_i) \Big)(x') = \sum_{i=1}^{k} f_i(x')((\phi(\tilde{S}_i))(x'))$$

Hence, ψ is continuous, since it is finite linear combination of continuous functions and this is true for all points $e' \in \Gamma(E_1)$. This completes the proof. □

Remark 4.28. *One can check that the result holds even for surjective maps, that is, if X is a normal space, and $(E_1, \pi_1), (E_2, \pi_2)$ be two vector bundles over the space X of rank k. Then there exists a surjective map $\phi : E_1 \to E_2$ if and only if there exists a surjective map $\psi : \Gamma(E_1) \to \Gamma(E_2)$. Similarly, it holds for injective maps as well.*

Remark 4.29. *We claim that trivial vector bundles over a compact Hausdorff space X are in one-to-one correspondence with the finitely generated free $C(X)$-modules. In order to see this, first note that any section of a trivial bundle of rank k is the set of continuous functions from X to $\mathbb{R}^k$, that is, the set $(C(X))^k$. Conversely, for a given finitely generated free $C(X)$-module $(C(X))^n$, we can find a trivial vector bundle, namely the trivial vector bundle of rank n over X which is unique up to isomorphism as proved earlier.*

4.5 Examples of Swan's theorem

Let us now see some examples of projective modules which are not free.

Example 4.30. *Let K be a field. Consider a ring $R := M_2(K)$,*

$$P := \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \; a, b \in K \right\}$$

Clearly, P is an R-module. Moreover, $R \cong P \oplus P$. Thus P is a projective module. Now, let us prove that P is not free. On contrary, let us assume that P is free, and let $\{x_i\}_{i \in I}$ be its basis. Then since $dim_K R = 4$ and $dim_R P = |I|$, we have $dim_K P = 4|I|$. But, by the definition of P, we also have $dim_K P = 2$ which implies, $2 = 4|I|$ which gives us a contradiction. Hence P is not free. We can extend this argument to $R = M_m(K)$ where, $m \geq 2, m \in \mathbb{N}$. Then each column space of R will be a projective module which is not a free R-module.

Lemma 4.31. *Let (E, π) be the Möbius bundle over $\mathbb{S}^1$, then $E \oplus E \cong \varepsilon^2$.*

Proof. Let us consider the trivializations to be $U = \mathbb{S}^1 - \{-1\}, V = \mathbb{S}^1 - \{1\}$, and corresponding transition functions to be as follows.

$$g_{UV}(e^{i\theta}) = \begin{cases} -I_2 & \text{if } 0 < \theta < \pi \\ I_2 & \text{if } \pi < \theta < 2\pi \end{cases}$$

Now, we can define $h_U : U \to GL_2(\mathbb{R})$ and $h_V : V \to GL_2(\mathbb{R})$ as follows.

$$h_U(e^{i\theta}) = \begin{cases} -I_2 & \text{if } 0 \leq \theta < \pi \\ I_2 & \text{if } \pi < \theta \leq 2\pi \end{cases}$$

and

$$h_V(e^{i\theta}) = I_2 \text{ for } \theta \in (0, 2\pi)$$

Then it can be easily checked that these functions are continuous and satisfy

$$h_U \cdot g_{UV} = g'_{UV} \cdot h_V \text{ on } U \cap V$$

,where g'_{UV} are constant identity matrices since they represent the transition data for trivial bundle ε^2. □

Example 4.32 (Möbius band)**.** *Let (E, π) denote the Möbius band, that is, the only non-trivial line bundle over $\mathbb{S}^1$. Consider the set of sections $\Gamma(E)$. From lemma 4.31 we know that $E \oplus E \cong \mathbb{S}^1 \times \mathbb{R}^2$. Hence, taking sections of both sides, we get $\Gamma(E)$ to be a projective $C(\mathbb{S}^1)$-module. From the remark 4.29 we know that there is one-to-one correspondence between trivial vector bundles over X and finite free $C(X)$-modules. Hence, $\Gamma(E)$ gives us an example of a projective module which is not free.*

Example 4.33. *Let $X = \mathbb{S}^2$, $E = T\mathbb{S}^2$ be the tangent bundle of $\mathbb{S}^2$. Then, by Hopf's theorem, that is, a generalization of Hairy ball theorem[8, 14] we know that $T\mathbb{S}^n$ is a trivial vector bundle if and only if n is odd, in particular $T\mathbb{S}^2$ is a non-trivial vector bundle. Moreover, we know that $TM \oplus N \cong \varepsilon|_{\mathbb{R}^k} \cong \varepsilon^k$, where M is a manifold embedded in $\mathbb{R}^k$, and N is the normal bundle which is the line bundle such that each fiber is spanned by the point itself. Taking sections of both the sides we get an example that $\Gamma(T\mathbb{S}^2)$ is a finitely generated projective $C(\mathbb{S}^2)$-module which is not free.*

Example 4.34. *Let us search for a non-compact space X which will give us an example of a vector bundle (E, π) for which the section $\Gamma(E)$ is not a finitely generated projective $C(X)$-module. Note that, if X is paracompact, Hausdorff, then we can not get an example out of it simply because, any vector bundle E of rank k over a paracompact, Hausdorff space is trivial bundle by corollary 5.27, which implies $\Gamma(E) \cong (C(X))^k$, that is, $\Gamma(E)$ is finitely generated projective module. One might try to get an example from proposition 4.21 by trying to construct an idempotent valued continuous function which is not the constant maps mapping to the identity $I_n \in M_n(\mathbb{R})$). Consider the topological space X to be $\mathbb{R}$. Consider $P : \mathbb{R} \to M_n(\mathbb{R})$ as the following map:*

$$P(\theta) = \begin{bmatrix} 1/2 + (\cos(\theta))/2 & (\sin(\theta))/2 \\ (\sin(\theta))/2 & 1/2 - (\cos(\theta))/2 \end{bmatrix}$$

Clearly, this is an idempotent valued continuous map which is not the trivial map. Note carefully that proposition 4.21 gives us a relation for same value of n, but it might very well happen that the function P fives a finitely generated projective $C(X)$-module which we get even after using the correspondence for some constant identity valued map to an identity $I_m \in M_m(\mathbb{R})$, where the correspondence is the one described in the proof of proposition 4.21. In other words, the correspondence described in proposition 4.21 is not a one-to-one correspondence. We can also look at it from geometric point of view. The corresponding vector bundle for P will be $Im(P)$ in which the range space is of rank 1 simply because it is not zero since P is not identically zero, and it is not 2 since the determinant of P is zero for all $\theta \in \mathbb{R}$. So, let $s(\theta)$ be a 2×1 column vector which generates range$(P(\theta))$. Then we can have a map $\phi : \varepsilon^1 \to Im(P)$ as $(t, v) \mapsto (t, v \cdot s(t))$ which can act as an isomorphism implying $Im(P) \cong \varepsilon^1$. One can consider the following function as $s(\theta)$ defined on $(0, 4\pi]$ as follows, and extended 4π periodically.

$$s(\theta) = \begin{cases} v_1(\theta) & \text{if } 0 < \theta \leq \pi/2 \\ v_2(\theta) & \text{if } \pi/2 < \theta \leq 3\pi/2 \\ -v_1(\theta) & \text{if } 3\pi/2 < \theta \leq 5\pi/2 \\ -v_2(\theta) & \text{if } 5\pi/2 < \theta \leq 7\pi/2 \\ v_1(\theta) & \text{if } 7\pi/2 < \theta \leq 4\pi \end{cases}$$

where, $v_1(\theta)$ is the first column of $P(\theta)$ and $v_2(\theta)$ is the second column of $P(\theta)$.

Remark 4.35. *The previous example shows us that compactness is not a necessary condition to get a one-to-one correspondence between finitely generated projective $C(X)$-modules and vector bundles over space X.*

Fact 4.36. *Let γ^1 be the tautological line bundle [3] over $\mathbb{RP}^\infty$. Then for any $k \in \mathbb{N}$, there does not exists any vector bundle η of rank k over $\mathbb{RP}^\infty$ such that $\gamma^1 \oplus \eta \cong \varepsilon^{k+1}$. Refer [15].*

Fact 4.37. *Direct limit of paracompact Hausdorff spaces is paracompact Hausdorff if the direct limit is compactly generated. Refer [2].*

Example 4.38. *We aim to find an example of a space which is not compact in a hope that the set of sections of a vector bundle over that space to violate finite generatedness. So,* *let $X = \mathbb{RP}^\infty$ with the direct limit topology where X_n's are $\mathbb{RP}^n$'s and F_{ij} are canonical injections obtained by adding zeros in the remaining co-ordinates. Consider γ^1 to be the tautological line bundle over $\mathbb{RP}^\infty$. Since $\mathbb{RP}^\infty$ is used a classifying space for principal bundles over $\mathbb{Z}_2$ [15], $\gamma^1 \ncong \varepsilon^1$ and we know that, since $\mathbb{RP}^n$ are a CW complexes, they are paracompact, Hausdorff space. So, using fact 4.37 we have $\mathbb{RP}^\infty$ to be paracompact, Hausdorff, hence it is normal so get that $\Gamma(\gamma^1) \ncong \Gamma(\varepsilon^1)$, that is, $\Gamma(\gamma^1)$ is not free. Now, if $\Gamma(\gamma^1)$ is finitely generated, then there exists a surjection from $C(\mathbb{RP}^\infty)^k \to \Gamma(\gamma^1)$ for some $k \in \mathbb{N}$. Hence, due to remark 4.28 we have a surjection from $\varepsilon^k \to \gamma^1$. Hence, using metric on ε^k we can write $\varepsilon^k \cong (\gamma^1) \oplus (\gamma^1)^\perp$. But using fact 4.36 we get a contradiction, hence $\Gamma(\gamma^1)$ can not be finitely generated. Note that, similar arguments can be used for $X = \mathbb{CP}^\infty$.*

5 Vector bundles: Kaplansky's theorem

5.1 Localization

Now we are going to study the algebraic analogues of a bundle being locally trivial, that is, localizations over $C(X)$ and when it is globally trivial.

Definition 5.1. *Let R be a commutative ring, $S \subseteq R$ be a multiplicatively closed set (that is, $1 \in S$ and $s, t \in S$ implies $s \cdot t \in S$). Then we define a ring called localization of R with respect to S and denote it by $S^{-1}R$ such that*

$$S^{-1}R := \left\{ \left[\frac{a}{b}\right] \middle| a \in R, b \in S \right\} \Big/ \sim$$

where $\frac{a_1}{b_1} \sim \frac{a_2}{b_2}$ if and only if there exists $t \in S$ such that $t(a_1b_2 - a_2b_1) = 0$.

$$\left[\frac{a_1}{b_1}\right] + \left[\frac{a_2}{b_2}\right] := \left[\frac{a_1b_2 + a_2b_1}{b_1b_2}\right], \left[\frac{a_1}{b_1}\right] \cdot \left[\frac{a_2}{b_2}\right] := \left[\frac{a_1a_2}{b_1b_2}\right]$$

Let us check that this addition and multiplication is well-defied, that is, numerator belongs to R, denominator belongs to S, and it respects the equivalence classes. The first two checks are immediate, for the third one, if

$$t_1, t_2 \in S \text{ such that } t_i(a_id_i - c_ib_i) = 0 \text{ for } i = 1, 2 \tag{8}$$

that is, $\left[\frac{a_i}{b_i}\right] = \left[\frac{c_i}{d_i}\right]$, for $i = 1, 2$. For addition, we have to show that $\left[\frac{a_1b_2 + a_2b_1}{b_1b_2}\right] = \left[\frac{c_1d_2 + c_2d_1}{d_1d_2}\right]$, that is, to show that there exists a $t \in S$ such that

$$t((a_1b_2 + a_2b_1)(d_1d_2) - (c_1d_2 + c_2d_1)(b_1b_2)) = 0.$$

Putting $t = t_1 \cdot t_2$ and rearranging, we get L.H.S. equals to

$$t_2b_2d_2(a_1d_1 - b_1c_1) + t_1b_1d_1(a_2d_2 - b_2c_2).$$

This is equal to zero due to 8. Now, for multiplication, we need to show that there exists $t \in S$ such that

$$t(a_1a_2d_1d_2 - b_1b_2c_1c_2) = 0. \tag{9}$$

From 8, by multiplying two equations with each other, we get

$$t_1t_2(a_1d_1a_2d_2 - a_1d_1b_2c_2 - b_1c_1a_2d_2 + b_1c_1b_2c_2) = 0.$$

That is,

$$t_1t_2(a_1d_1a_2d_2 - a_1d_1b_2c_2 - b_1c_1a_2d_2 + 2b_1c_1b_2c_2 - b_1c_1b_2c_2) = 0.$$

After rearranging the terms, we get

$$t_1t_2(a_1d_1a_2d_2) + b_2c_2t_2t_1(b_1c_1 - a_1d_1) + b_1c_1t_1t_2(b_2c_2 - a_2d_2) = 0.$$

Again using 8 we get $t_1t_2(a_1d_1a_2d_2 - b_1b_2c_1c_2) = 0$, that is, 9 gets satisfied for $t = t_1 \cdot t_2$, hence we are done.

Example 5.2. *If $\mathfrak{p}$ is a prime ideal, then $S := R - \mathfrak{p}$ is a multiplicatively closed set simply because, $1 \in S$ as $1 \notin \mathfrak{p}$; $s, t \in S$ implies $s, t \notin \mathfrak{p}$ which in turn implies $s \cdot t \notin \mathfrak{p}$ because, if $s \cdot t \in \mathfrak{p}$ then since $\mathfrak{p}$ is prime, either $s \in \mathfrak{p}$ or $t \in \mathfrak{p}$ or both which can not be true by our assumption. Hence, $s \cdot t \in S$. Since maximal ideals are prime ideals, so we will denote the localizations of ring R due to the maximal ideals of the form $\mathfrak{m}_x$ as R_x. Moreover, we also claim the following.*

Lemma 5.3. *Let $\mathfrak{p}$ be a prime ideal, then the localization constructed by using $S = R - \mathfrak{p}$ is a local ring.*

Proof. Denote the localization by $R_{\mathfrak{p}}$, then we claim that the only maximal ideal in $R_{\mathfrak{p}}$ is $\mathfrak{p}R_{\mathfrak{p}} := \left\{ \left[\frac{a}{b}\right] \middle| a \in \mathfrak{p}, b \in R - \mathfrak{p} \right\}$. Clearly, $\mathfrak{p}R_{\mathfrak{p}}$ is an ideal since $a_1 + a_2 \in \mathfrak{p}R_{\mathfrak{p}}$ for all $a_1, a_2 \in \mathfrak{p}R_{\mathfrak{p}}$ and $a \cdot x \in \mathfrak{p}R_{\mathfrak{p}}$ for all $a \in \mathfrak{p}R_{\mathfrak{p}}, x \in R_{\mathfrak{p}}$, as $\mathfrak{p}$ itself is an ideal. Now, we will show that, if $I \not\subseteq \mathfrak{p}R_{\mathfrak{p}}$ is an ideal, then I is the full ring $R_{\mathfrak{p}}$. For then, it will not only imply that $\mathfrak{p}R_{\mathfrak{p}}$ is a maximal ideal, but also that $\mathfrak{p}R_{\mathfrak{p}}$ is the only maximal ideal. So, let $\left[\frac{a}{b}\right] \in I$ such that $a \notin \mathfrak{p}$, that is, $a \in R - \mathfrak{p}$. Then $\left[\frac{b}{a}\right] \in R$ and I being an ideal $\left[\frac{a}{b}\right] \cdot \left[\frac{b}{a}\right] \in I$ which implies $1 \in I$, hence proved. □

Definition 5.4. *Let M be an R-module, $S \subseteq R$ be a multiplicatively closed set. Then we define a module over $S^{-1}R$ called localization of M with respect to S and denote it by $S^{-1}M$ such that*

$$S^{-1}M := \left\{ \left[\frac{m}{s}\right] \middle| m \in M, s \in S \right\} \Big/ \sim$$

where $\frac{m_1}{s_1} \sim \frac{m_2}{s_2} \Leftrightarrow$ there exists $t \in S$ such that $t(s_2 m_1 - s_1 m_2) = 0$.

$$\left[\frac{m_1}{s_1}\right] + \left[\frac{m_2}{s_2}\right] := \left[\frac{s_2 m_1 + s_1 m_2}{s_1 s_2}\right], \left[\frac{a}{s_1}\right]\left[\frac{m}{s_2}\right] := \left[\frac{am}{s_1 s_2}\right], \text{where } a \in R$$

To check that this addition and R-action is well-defied, note that the numerators belong to M, and denominators belong to S. To show that they respect the equivalence classes, the proof for addition is similar as that for localization over a ring.Now, for group action, let $\left[\frac{m_1}{s_1}\right]$ and $\left[\frac{m_2}{s_2}\right]$ be such that

$$t_1(s_2 m_1 - s_1 m_2) = 0 \text{ for some } t_1 \in S. \tag{10}$$

and $\left[\frac{a_1}{s_1'}\right]$ and $\left[\frac{a_2}{s_2'}\right]$ be such that

$$t_2(s_2' a_1 - s_1' a_2) = 0 \text{ for some } t_2 \in S. \tag{11}$$

We need to prove that $\left[\frac{a_1 m_1}{s_1' s_1}\right] = \left[\frac{a_2 m_2}{s_2' s_2}\right]$, that is,

$$t(s_2' s_2 a_1 m_1 - s_1' s_1 a_2 m_2) = 0 \text{ for some } t \in S. \tag{12}$$

Multiplying equation 10 and 11 we get

$$t_1t_2(s_2s_2'a_1m_1 - s_2's_1a_1m_2 - s_1's_2a_2m_1 + s_1's_1a_2m_2) = 0.$$

Adding and subtracting $s_1's_1a_2m_2$ once, we get

$$t_1t_2(s_2s_2'a_1m_1 - s_2's_1a_1m_2 - s_1's_2a_2m_1 + 2s_1's_1a_2m_2 - s_1's_1a_2m_2) = 0.$$

Rearranging all the terms gives us the following equation.

$$t_1t_2(s_2's_2a_1m_1 - s_1's_1a_2m_2) - s_1m_2t_1t_2(s_2'a_1 - s_1'a_2) - s_1'a_2t_2t_1(s_2m_1 - s_1m_2) = 0.$$

Now, invoking 10 and 11 again, we get $t = t_1t_2$ for which 12 gets satisfied. Hence the R-action is also well-defined.

Example 5.5. *If $\mathfrak{p}$ is a prime ideal, then we can have a multiplicative set $S = R - \mathfrak{p}$ similar to the example 5.2. Again, since maximal ideals are prime ideals, we will denote the localizations of maximal ideals of the form $\mathfrak{m}_x$ over modules P as P_x.*

5.2 Application of localization of modules for vector bundles

Lemma 5.6. *Let R be a commutative ring, P be a finite rank projective R-module, $S \subseteq R$ be a multiplicatively closed set, then $S^{-1}P$ is also a finite rank projective $S^{-1}R$-module.*

Proof. In order to prove this, it is enough to prove that $S^{-1}(P \oplus Q) \cong S^{-1}P \oplus S^{-1}Q$. For then, since P is projective, we have an R-module Q such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. Then taking localization on both the sides, we get $S^{-1}P \oplus S^{-1}Q \cong S^{-1}(R^n) \cong (S^{-1}R)^n$.

Now, consider an element $\left[\dfrac{p}{s_1}\right] \in S^{-1}P$, $\left[\dfrac{q}{s_2}\right] \in S^{-1}Q$ and map this to $\left[\dfrac{(s_2 \cdot p + s_1 \cdot q)}{s_1 \cdot s_2}\right]$ under a map say, $\Phi : S^{-1}P \oplus S^{-1}Q \to S^{-1}(P \oplus Q)$. It is easy to check that this is a $S^{-1}R$-linear map. Also, it indeed is well-defined. So, it remains to prove that Φ is a bijection. Let $\left[\dfrac{(p+q)}{s}\right] \in S^{-1}(P \oplus Q)$ then $\Phi\left(\left[\dfrac{p}{s}\right] + \left[\dfrac{q}{s}\right]\right) = \left[\dfrac{(p+q)}{s}\right]$ simply because $\Phi\left(\left[\dfrac{p}{s}\right] + \left[\dfrac{q}{s}\right]\right) = \left[\dfrac{s \cdot p + s \cdot q}{S^2}\right] = \left[\dfrac{(p+q)}{s}\right]$ since for $1 \in S$ we have $1(s(s \cdot p + s \cdot q) - S^2(p+q)) = 0$. Hence, Φ is surjective. Now, let $\Phi\left(\left[\dfrac{p}{s_1}\right] + \left[\dfrac{q}{s_2}\right]\right) = 0$, that is, $\left[\dfrac{p+q}{s_1 \cdot s_2}\right] = 0$, that is, $p + q = 0$ which implies $p = 0 = q$. Hence $\left(\left[\dfrac{p}{s_1}\right] + \left[\dfrac{q}{s_2}\right]\right) = 0$. This proves the injevtivity and hence Φ is an isomorphism and this completes the proof. ☐

Consider X to be a compact Hausdorff space, $R = C(X)$ and a maximal ideal of the form $\mathfrak{m}_x$ for some $x \in X$. Then we can talk about $C(X)_x$ since a maximal ideal is a prime ideal.

Let $\mathcal{P}|_R$ be the set of all finitely generated projective R-modules up to isomorphism, and let $Vect(X)$ denote the set of all vector bundles over base space X up to isomorphism. Note that $Vect(X)$ is a disjoint union of $Vect^k(X), k \geq 0$, where $Vect^k(X)$ is the set of all rank k vector bundles over base space X up to isomorphism.

Definition 5.7 (Direct limit)**.** *Let I be a directed, that is, totally ordered set. A directed system of spaces, indexed by I, is a collection of spaces X_i with continuous maps $f_{ji} : X_i \to X_j$ if $i \leq j$ such that $f_{ii} = Id_{X_i}$ and $f_{kj} \circ f_{ji} = f_{ki}$. The direct limit space is defined to be the quotient space*

$$\varinjlim X_j := \Big(\bigsqcup_{j \in I} X_j\Big) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by the identifications $X_i \ni x \sim f_{ji}(x) \in X_j$ for $i \leq j$ according to the order of I.

Definition 5.8. *Let X be a topological space, $x \in X$ be a point. Then we define germ of vector bundle at x as the set $\{[(E, U)] | E \in Vect(X), U$ is an open set containing $p\}$ where, $(E_1, U_1) \sim (E_2, U_2)$ if $E_1|_V \cong E_2|_V$ for some open set $V \subseteq U_1 \cap U_2$. Denote it by $germ_x$.*

Now, we claim that, for each $x \in X$ the following diagram commutes.

$$\begin{array}{ccc} Vect(X) & \xrightarrow{\Phi_1} & germ_x \\ \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_3} \\ \mathcal{P}|_{C(X)} & \xrightarrow{\Phi_4} & \mathcal{P}|_{C(X)_x} \end{array}$$

All four maps are set theoretic maps which respects the isomorphism classes of each object. The maps are described below.

- Let $(E, \pi) \in Vect(X)$ and let $\{U_i\}_{i \in I}$ be trivializing neighbourhoods. Then map $E \mapsto [(E, U)]$ where U is a trivializing neighbourhood containing x. In order to check that this map is well-defined, consider the case when x lies in multiple trivializing neighbourhoods, say U_1, U_2, then clearly $(E, U_1) \sim (E, U_2)$ because we have an open set $V \subseteq U_1 \cap U_2$, namely, $V = U_1 \cap U_2$ such that $E|_V = E|_V$ is tautologically true. We also need to check if $(E_1, \pi_1), (E_2, \pi_2)$ are isomorphic bundles of rank k, then they map to the same element in $germ_x$. We can have a common refinement of trivializing neighbourhoods which cover X for both E_1 and E_2. Let U be a trivializing neighbourhood which contains x. Note that E_1 and E_2 get mapped to (E_1, U) and (E_2, U) respectively under Φ_1. In order to show that they are equivalent, consider $V = U$ itself, and observe that $E_1|_V \cong U \times \mathbb{R}^k \cong E_2|_V$. Hence Φ_1 is indeed well-defined.
- Φ_2 is studied explicitly in the proof of Swan's theorem.

- Let E be a vector bundle of rank k, and U be an open set containing x. Let $\{(E_{\alpha,U_\alpha})\}_{\alpha\in\Lambda}$ be members of the equivalence class $[(E,U)]$. Let (E_α, U_α) be fixed. Consider $\{V_i^\alpha\}_{i\in I}$ be trivializing neighbourhoods of E_α which cover X. Then there exists an open set in $\{V_i^\alpha\}_{i\in I}$ which contains x. Call that set V_α. Then note that there exists an open set W_α such that $(E_\alpha, U_\alpha) \sim (\varepsilon^k, W_\alpha)$, namely, $W_\alpha = U_\alpha \cap V_\alpha$. Then, having the collection $\{(E_\alpha, U_\alpha)\}_{\alpha\in\Lambda}$ is equivalent of having the collection $\{(\varepsilon^k, W_\alpha)\}_{\alpha\in\Lambda}$. Now, denote the set of sections over ε^k restricted to W_α as $\Gamma(E_\alpha)$. Since the map Φ_2 maps a vector bundle to its set of sections, we are searching for Φ_3 to be of similar kind. Note that, if we had $\{W_\alpha\}_{\alpha\in\Lambda}$ to be a totally ordered set, that is, for any given $\beta, \gamma \in \Lambda$, either $W_\beta \subseteq W_\gamma$ or $W_\gamma \subseteq W_\beta$, without loss of generality, if we had $W_\beta \subseteq W_\gamma$, then we could have defined a function $f_{\beta\gamma} : \Gamma(E_\gamma) \to (\Gamma(E_\beta))$ as $(s : w_\gamma \to \varepsilon^k) \mapsto (s|_{W_\beta} : W_\beta \to \varepsilon^k)$. Note that this certainly satisfies the condition mentioned in the definition of direct limit (5.7), that is, $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$. So, we can define the direct limit of $\Gamma(E_\alpha)$ and denote it by $\varinjlim \Gamma(E_\alpha)$. In a more general setting, it is also called stalk of a sheaf at the point x. So, let us try to obtain a totally ordered set which is in principle equivalent to $\{(\varepsilon^k, W_\alpha)\}_{\alpha\in\Lambda}$. To do this, consider two elements W_α, W_β, if either of them is contained in other, then we are done. Otherwise, redefine one of these sets, say W_α as $W'_\alpha = W_\alpha \cap W_\beta$, and keep W_β as it is. Clearly, $(\varepsilon^k, W_\alpha) \sim (\varepsilon^k, W_\alpha \cap W_\beta) = (\varepsilon^k, W'_\alpha)$, so nothing is really changing. Now, note that X is compact, hence this process of redefining sets will finish in at most finitely many steps and give us the required totally ordered collection. Now it remains to prove (i) $\varinjlim \Gamma(E_\alpha) \in \mathcal{P}|_{C(X)_x}$, that is, $\varinjlim \Gamma(E_\alpha)$ is a finitely generated projective $C(X)_x$-module, and (ii) $\varinjlim \Gamma(E_\alpha) \cong (C(X)_x)^k$. Though, it is enough to prove (ii) since (ii) trivially implies (i). To prove (ii), note that, each $\Gamma(E_\alpha)$ realized as set of sections of ε^k over W_α has a k-tuple of linearly independent sections, which do not vanish on W_α since they are linearly independent, in particular, they do not vanish at x which gives us (ii).

- Using definition 5.4 we can map a finitely generated projective module P to P_x under the map Φ_4. Moreover, using lemma 5.6 and lemma 4.20 we get that the map Φ_4 is indeed well-defined.

To check the commutativity of the diagram, let $E \in Vect^k(X)$, then $\Phi_3 \circ \Phi_1(E) = (C(X)_x)^k$. So, it remains to prove that $\Phi_4(\Gamma(E)) = (C(X)_x)^k$. To prove this, let $\left[\frac{m}{s}\right] \in \Phi_4(\Gamma(E))$, that is, $m \in \Gamma(E), s \in C(X) - m_x$, that is, $s(x) \neq 0$. Now, from proposition 4.9 we can write $m = \sum_{i=1}^{k} f_i \cdot s_i$, where f_i and s_i are defined over U. Consider V to be an open set such that $x \in V \subseteq \overline{V} \subseteq U$ and using Urysohn's lemma let $g : X \to [0,1]$ be a function which is 1 on $\overline{V}$, zero on U^C. Obtain $\tilde{s_i} := s_i \cdot g$, which is continuous and defined on whole of X. Now consider $m' := \sum_{i=1}^{k} f_i \cdot \tilde{s_i}$.

Now we claim that $\left[\frac{m}{s}\right] = \left[\frac{m'}{s}\right]$. So consider a function $t \in C(X)$ such that $t(x) = 1, t(V^C) = \{0\}$, which implies $t(s(m-m')) = 0$ since t is zero outside V and $m = m'$ on V. In order to conclude the proof, see that m' can be written as direct sum of $\left\{\left[\frac{\tilde{s}_i}{s}\right]\right\}_{i=1}^{k}$ where $\left[\frac{\tilde{s}_i}{s}\right] \in C(X)_x$ since $\tilde{s}_i(x) \neq 0$ for all x because $\tilde{s}_i = s_i \cdot g$ and $g(x) \neq 0$ by construction, and $s_i(x)$ can not be zero since they are basis elements for E_x. This proves that $\Phi_3 \circ \Phi_1 = \Phi_4 \circ \Phi_2$, that is, the diagram commutes.

From the commutativity of the diagram, we see that the image of finitely generated projective modules which are also finitely generated projective over the local ring $C(X)_x$ are free. So one can naturally ask if this is true for any local ring in general. The answer is yes. The proof is given in 5.11 which follows from the Nakayama's lemma (A.5). Moreover, one can relax the condition of finitely generatedness and still get the modules to be free. This is known as Kaplansky's theorem which will be proved in the next subsection.

Lemma 5.9. *Let R be a commutative ring, M be an R-module, I be an ideal in R, then M/IM is a module over R/I.*

Proof. Clearly, IM is a submodule of M since $a \cdot \sum_{i=1}^{n} a_i m_i = (a \cdot a_i) m_i \in IM$ because I is an ideal, that is, $a \cdot a_i \in I$ for all $a_i \in I, a \in R$ and closure under addition is also trivial. Consider, $M/IM = \{m + IM \mid m \in M\}$. Define an R/I action as follows:

$$(a + I) \cdot (m + IM) := (a \cdot m + IM)$$

Let us check that this action is well-defined. So, let $a_1, a_2 \in R$ such that $a_1 - a_2 \in I$, $m_1, m_2 \in M$ such that $m_1 - m_2 \in IM$. Then we need to prove that $a_1 m_1 - a_2 m_2 \in IM$. Consider, $(a_1 - a_2)(m_1 - m_2) = a_1 m_1 + a_2 m_2 - a_2 m_1 - a_1 m_2$. Adding and subtracting $a_2 m_2$ once, and rearranging, we get

$$a_1 m_1 - a_2 m_2 = (a_1 - a_2)(m_1 - m_1) - a_2(m_2 - m_1) - (a_2 - a_1) m_2.$$

Note that $(a_1 - a_2) \in I$, hence $(a_1 - a_2)(m_1 - m_2) \in IM$ and $(a_2 - a_1) m_2 \in IM$. Since $(m_2 - m_1)$ can be written as $\sum_{i=1}^{n} r_i n_i$, where $r_i \in I, n_i \in M$ for all $i \in \{1, \cdots, n\}$. Now, since I is an ideal, $\sum_{i=1}^{n} (a_2 \cdot b_i) n_i \in IM$ as $a_2 \cdot b_i \in I$ for all $a_2 \in R, b_i \in I$. Hence, the action is well-defined. Other properties are also satisfied due to the fact that M is an R-module. ☐

Lemma 5.10. *Let R be a ring, I be an ideal of R and A, B, C be R-modules such that $A \cong B \oplus C$, then $A/IA \cong B/IB \oplus C/IC$.*

Proof. Consider an element $(a + IA) \in A/IA$, that is, $a \in A$. Now decompose a uniquely as $a = b + c$, where $b \in B$ and $c \in C$. Map $(a + IA)$ to $(b + IB) + (c + IC)$.

Note that this decomposition is indeed unique, and possible for all $(a+IA) \in A/IA$. So, it remains to prove that it is well-defined. Let $a_1, a_2 \in A$ such that $(a_1 - a_2) \in IA$, we need to show that the corresponding $(b_1 - b_2) \in IB$ and $(c_1 - c_2) \in IC$. Since $(a_1 - a_2) \in IA$, we can write $(a_1 - a_2) = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \in R, x_i \in A$ for all $i \in \{1, \cdots, n\}$. Decompose x_i's uniquely as $y_i + z_i$ where $y_i \in B, z_i \in C$ for all $i \in \{1, \cdots, n\}$. Hence, we can write

$$b_1 - b_2 + c_1 - c_2 = a_1 - a_2 = \sum_{i=1}^{n} \lambda_i y_i + \sum_{i=1}^{n} \lambda_i z_i.$$

Since $B \cap C = \{0\}$, we get $(b_1 - b_2) = \sum_{i=1}^{n} \lambda_i y_i, (c_1 - c_2) = \sum_{i=1}^{n} \lambda_i z_i$, that is, $(b_1 - b_2) \in IB, (c_1 - c_2) \in IC$. This completes the proof. □

Proposition 5.11. *Finitely generated projective modules over commutative local rings are free.*

Proof. Let R be a commutative local ring with $\mathfrak{m}$ as the maximal ideal. Let M be a finitely generated projective R-module. Since $\mathfrak{m}$ is a maximal ideal, $R/\mathfrak{m}$ is a field $\Bbbk$. From lemma 5.9 we can consider $M/\mathfrak{m}M$ to be a module over $R/\mathfrak{m}$, but since $R/\mathfrak{m}$ is a field, we get $M/\mathfrak{m}M$ as a vector space over $\Bbbk$. Hence, we can talk about its dimension. Since M is finitely generated, the dimension should also be finite, say n. Now, let $\{m_i + \mathfrak{m}M\}_{i=1}^{n}$ be the generators of $M/\mathfrak{m}M$. Then using a version of Nakayama's lemma, that is, proposition A.7 we can say that $\{m_1, \cdots, m_n\}$ generate M as an R-module. Now, consider the map $\phi : R^n \to M$ as $a_i \mapsto m_i$, where a_i's are generators, namely, $a_i = (r_1, \cdots, r_n)$ such that $r_i = 1, r_j = 0$ for all $j \neq i$. Then ϕ is a surjective map. Hence, it is enough to prove that $ker(\phi) = 0$. Moreover, we can have the following short exact sequence.

$$0 \longrightarrow ker(\phi) \hookrightarrow R^n \longrightarrow M \longrightarrow 0$$

Hence, using proposition 4.19 and the fact that M is projective, the short exact sequence splits and we get $R^n \cong ker(\phi) \oplus M$ as R-modules. Now, using 5.10 we get

$$R^n/\mathfrak{m}R^n \cong ker(\phi)/\mathfrak{m}ker(\phi) \oplus M/\mathfrak{m}M.$$

Now, note that from 5.10 we also get $R^n/\mathfrak{m}R^n \cong \bigoplus_{i=1}^{n} R/\mathfrak{m}R \cong \Bbbk^n$, and on the right hand side, $M/\mathfrak{m}M \cong \Bbbk^n$ which makes $ker(\phi)/\mathfrak{m}ker(\phi) = 0$, that is, $ker(\phi) = \mathfrak{m}ker(\phi)$, hence using the most familiar version of Nakayama's lemma, that is, lemma A.5 we get $ker(\phi) = 0$. This completes the proof. □

5.3 Kaplansky's theorem

We will refer [16] for this entire subsection.

Lemma 5.12. *If R is a local ring and $x \in R$, then either x is a unit or $(1-x)$ is a unit.*

Proof. Let $\mathfrak{m}$ be the maximal ideal, and let $x \in R$ be an element.
Case I: If $x \notin \mathfrak{m}$, then we claim that x is a unit. If x is not a unit, then consider the ideal generated by x, that is, $\langle x \rangle$. By Zorn's lemma, there exists a maximal ideal containing $\langle x \rangle$ [1]. Now, since there is a unique maximal ideal $\mathfrak{m}$, we have $\langle x \rangle \subseteq \mathfrak{m}$ which implies $x \in \mathfrak{m}$ which gives us a contradiction. Hence x is a unit.
Case II: If $x \in \mathfrak{m}$, then we claim that $(1-x) \notin \mathfrak{m}$ which implies $(1-x)$ is a unit using case I. So, let if possible $(1-x) \in \mathfrak{m}$, then $(1-x)+x \in \mathfrak{m}$, since an ideal is closed under addition. This implies $1 \in \mathfrak{m}$ but this is a contradiction since a maximal ideal is a proper ideal by definition. Hence proved. □

Lemma 5.13. *Let R be a commutative, local ring, let M be a matrix whose diagonal entries are units and off-diagonal entries are non-units, then M is invertible.*

Proof. Let $M = (m_{ij})_{n\times n}$ be a matrix whose diagonal entries are units and off-diagonal entries are non-units. Since R is a commutative ring, it is enough to prove that the determinant of M is a unit. Apply row operations as follows $R_1 \times m_{11}^{-1}, R_j - m_{j1} \times R_1$ for all $j \in \{2, 3, \cdots, n\}$.

$$\begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & m'_{12} & \dots & m'_{1n} \\ 0 & m'_{22} & \dots & m'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & m'_{n2} & \dots & m'_{nn} \end{bmatrix}$$

Let us assume that these row operations gives a matrix called M_1. We claim that m'_{22} is also a unit, for then applying $R_2 \times {m'_{22}}^{-1}, R_j - m_{j2} \times R_2$ for all $j \in \{3, 4, \cdots, n\}$ on M_1 will give us M_2 and so on. Note that $det(M) = m_{11} \cdot det(M_1)$. So, applying the row operations n times gives us M_n whose which is an upper triangular matrix with diagonal entries equal to one. So, using $det(M) = m_{11} \cdot det(M_1)$ recursively we will get the result. So, now it remains to prove that m'_{22} is a unit. We have $m'_{22} = m_{22} - m_{11}^{-1} \cdot m_{12} \cdot m_{21}$, that is, $m_{22} = m_{11}^{-1} \cdot m_{12} \cdot m_{21} + m'_{22}$. Multiplying both the sides by m_{22}^{-1}, we get $1 = m_{11}^{-1} \cdot m_{22}^{-1} \cdot m_{12} \cdot m_{21} + m_{22}^{-1} \cdot m'_{22}$. Now, let if possible, m'_{22} is not a unit, then so is $m_{22}^{-1} \cdot m'_{22}$. So by lemma 5.12 we have $1 - m_{22}^{-1} \cdot m'_{22}$ is a unit, that is, $m_{11}^{-1} \cdot m_{22}^{-1} \cdot m_{12} \cdot m_{21}$ is a unit, but this is a contradiction since m_{12} and m_{21} are non-units. This completes the proof. □

Remark 5.14. *One can not expect the lemma 5.13 to be true for any commutative ring in general. To see this, consider $\mathbb{Z}$ to be a commutative ring, and consider the following matrix.*

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

This is a matrix with diagonal entries to be units and off-diagonal entries to be non-units, but still it does not have inverse in $M_2(\mathbb{Z})$ since, if there was an inverse then it would be its inverse also in $M_2(\mathbb{R})$. But its inverse in $M_2(\mathbb{R})$ is

$$\begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}$$

which does not lie in $M_2(\mathbb{Z})$. Moreover, inverses are unique, hence A is not invertible in $M_2(\mathbb{Z})$.

Definition 5.15. *An R-module A is called countably generated, if there exists a surjection $\phi : \bigoplus_{i \in I} A_i^{I_i} \to A$, where A_i's are R-modules and $A_i^{I_i} = \bigoplus_{j \in I_i} A_i$, $A_i \neq A_j$ for $i \neq j$ and I is a countable set.*

Theorem 5.16 (Kaplansky)**.** *Projective modules over local rings are free.*

Proof. In order to prove this result, we will use several lemmas as follows.

Lemma 5.17. *If M is a countably generated R-module and P is a direct summand of M (that is, $M \cong P \oplus Q$ for some R-module Q), then P is also coutably generated.*

Proof. Let $\phi : \bigoplus_{i \in I} A_i^{I_i} \to M$ be a surjection. Then we get a surjection $pr_1 \circ \phi : \bigoplus_{i \in I} A_i^{I_i} \to P$ such that I is a countable set. $\square$

Corollary 5.18. *If P is a projective R-module, then P is countably generated.*

Proof. We can consider $M = \bigoplus_{i \in I} R$ which is countably generated by definition and so is P because, $M \cong P \oplus Q$ for some R-module Q. $\square$

Lemma 5.19. *Let R be a ring, M be a countably generated R-module. Suppose any direct summand N of M has following property: if $x \in N$, then there exists a free summand F of N such that $x \in F$, then M is free.*

Proof. Let $x_1, x_2, \cdots$ be generators of M, that is, $x_i \in A_i^{I_i}$ for all $i \in I$. Since,$A_i^{I_i}$ is a direct summand of M, there exists a free module F_i such that $x_i \in F_i$. So, since $\{x_i\}_{i \in \mathbb{N}}$ span M, we get $M = \bigoplus_{i \in \mathbb{N}} F_i$ which is free since each F_i is free. $\square$

So, in order to prove the theorem, it is enough to prove the following lemma.

Lemma 5.20. *Let P be a projective module of a local ring R. Then any element of P is contained in a free direct summand of P.*

Proof. Since P is projective, there exists a free projective module F such that $F = P \oplus Q$ for some R-module Q. Let $x \in P$, let B be a basis of F (that is, any element $k \in F$ can be written uniquely as $k = \sum_{i=1}^{n} a_i e_i, a_i \in R, e_i \in B$). Let B be such that the number of basis elements required to represent x is minimal (that is, B depends on the choice of x). Hence, if $x = \sum_{i=1}^{n} a_i e_i$, then no a_j can be expressed as linear combination of other a_i's, because, if $a_j = \sum_{i \neq j} a_i b_i$, then replacing e_i by $e_i + b_i e_j$ will leave the sum unchanged, since

$$x = \sum_{i \neq j} a_i(e_i + b_i e_j) = \sum_{i \neq j} (a_i e_i + a_i b_i e_j) = \sum_{i \neq j} a_i e_i + \Big(\sum_{i \neq j} a_i b_i\Big) e_j = \sum_{i=1}^{n} a_i e_i = x$$

But $\sum_{i \neq j} a_i(e_i + b_i e_j)$ has a shorter expression which contradicts the minimality.
Now, let $e_i = y_i + z_i$ such that $y_i \in P, z_i \in Q$. Write

$$y_i = \sum_{j=1}^{n} b_{ij} e_j + t_i \tag{13}$$

where t_i is linear combination of terms in B other than $\{e_1, e_2, \cdots, e_n\}$. To finish the proof it suffices to prove that the matrix $(b_{ij})_{n \times n}$ is invertible. For then, the map $\Psi : F \to F$ mapping $e_i \to y_i$ for all $i \in \{1, 2, \cdots, n\}$ and fixing $B - \{e_1, e_2, \cdots, e_n\}$ forms an isomorphism. So, $\{y_1, y_2, \cdots, y_n\}$ together with $B - \{e_1, e_2, \cdots, e_n\}$ forms a basis for F. Then define a submodule $N = span\{y_1, y_2, \cdots, y_n\}$ which is free and submodule of P. We have

$$x = \sum_{i=1}^{n} a_i e_i = \sum_{i=1}^{n} a_i(y_i + z_i) = \sum_{i=1}^{n} a_i y_i + \sum_{i=1}^{n} a_i z_i$$

Now, since Ψ is an isomorphism, we can write $e_i = \sum_{j=1}^{n} c_{ij} y_i$ where $i \in \{1, 2, \cdots, n\}$.
Substituting this in the equation above, we get

$$\sum_{i=1}^{n} \Big(a_i\Big(\sum_{j=1}^{n} c_{ij} y_j\Big)\Big) - \sum_{i=1}^{n} a_i y_i = \sum_{i=1}^{n} a_i z_i$$

Now, since $P \cap Q = \{0\}$, we get $\sum_{i=1}^{n} a_i z_i = 0$, hence

$$x = \sum_{i=1}^{n} a_i e_i = \sum_{i=1}^{n} a_i y_i \in N \tag{14}$$

Now, let us prove that $(b_{ij})_{n\times n}$ is invertible. Using 13 and 14 and equating the coefficients of e_j's, we get $a_j = \sum_{i=1}^{n} a_i b_{ij}$. But, since a_j can not be written as a linear combination of other a_i's, b_{ij} is non-unit for all $i \neq j$ and $(1 - b_{ii})$ is also non-unit. Since $(1 - b_{ii})$ is a non-unit, from lemma 5.12, $1 - (1 - b_{ii}) = b_{ii}$ is a unit. Now, using lemma 5.13 we get that the matrix $(b_{ij})_{n\times n}$ is invertible. □

This completes the proof of the Kaplansky's result. □

One can naturally ask if we can use this Kaplansky's result for any $C(X)$. The answer unfortunately is, for a huge class of spaces, that is, when X is normal, we can not have $C(X)$ to be a local ring , simply because for each $x, y \in X, x \neq y$, there exists functions $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$ since singletons $\{x\}, \{y\}$ are closed by definition of X being normal. Hence, f is not a unit since $f(x) = 0$, and $(1 - f)$ is also a non-unit since $(1 - f)(y) = 0$, hence by lemma 5.12 $C(X)$ can not be a local ring.

Remark 5.21. *As we have seen Φ_2 is in fact a bijection, similarly, Φ_3 is also a bijection. To prove this, let us first consider $[(E_1, U_1)], [(E_2, U_2)]$ such that $\Phi_3([(E_1, U_1)]) = \Phi_3([(E_2, U_2)])$. Again, from lemma 3.21 we can have a common refinement of the trivializing neighbourhoods of E_1 and E_2, also, without loss of generality we can assume U_1 and U_2 to be elements in the common refinement. So, we can consider an open set $V = U_1 \cap U_2$ which is non-empty since $x \in U_1, U_2$. Now, since $E_1|_V$ and $E_2|_V$ are both trivial vector bundles over a same set V, and of same rank k, we can say that $E_1|_V \cong E_2|_V$, that is, $[(E_1, U_1)] = [(E_2, U_2)]$. This proves that Φ_3 is injective. Now, for surjectivity, by Kaplansky's result, any projective module over $C(X)_x$ is of the form $\bigoplus_{i\in I} C(X)_x$ and finitely generatedness implies that $|I|$ is finite, say $|I| = n$, then there exists an element in $germ_x$, namely $[(\varepsilon^n, X)]$ which maps to $(C(X)_x)^n$ under Φ_3.*

5.4 Globally trivial vector bundles

Definition 5.22. *Let $f : X \to Z, g : Y \to Z$ be two continuous maps then define*

$$X \times_Z Y := \{(x, y) \in X \times Y | f(x) = g(y)\}$$

as a pullback of $g : Y \to Z$ by f. Sometimes, $X \times_Z Y$ will be denoted by $f^*(Y)$. In other words, the following diagram commutes.

$$\begin{array}{ccc} f^*(Y) & \xrightarrow{pr_2} & Y \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

Lemma 5.23. *Let (E, π) be a vector bundle of rank k over $X \times [0, 1]$. If $E|_{X\times[a,b]}$ and $E|_{X\times[b,c]}$ are trivial, then $E|_{X\times[a,c]}$ is trivial.*

Proof. Since $E|_{X\times[a,b]}$ and $E|_{X\times[b,c]}$ are trivial, we have the following trivializations.

$$\Phi_1 : E|_{X\times[a,b]} \to X \times [a,b] \times \mathbb{R}^k, \Phi_2 : E|_{X\times[b,c]} \to X \times [b,c] \times \mathbb{R}^k$$

Also consider the maps which we get after restricting Φ_i's to $X \times \{b\}$.

$$\phi_1 : E|_{X\times\{b\}} \to X \times \{b\} \times \mathbb{R}^k, \phi_2 : E|_{X\times\{b\}} \to X \times \{b\} \times \mathbb{R}^k$$

Note that ϕ_i's are homeomorphisms hence if we keep Φ_2 as it is and replace Φ_1 by $\phi_2 \circ \phi_1^{-1} \circ \Phi_1$ we get a trivialization of whole of $X \times [a,c]$. Hence $E|_{X\times[a,c]}$ is trivial. □

Lemma 5.24. *Let (E,π) be a vector bundle of rank k over $X \times [0,1]$ then there exists an open cover $\{U_i\}_{i\in I}$ of X such that $E|_{U_i\times[0,1]} \to (U_i \times [0,1]) \times \mathbb{R}^k$ is trivial.*

Proof. Let $x \in X$ be a point. Consider a compact set $\{x\} \times [0,1]$. Now, for each point $(x,t) \in \{x\} \times [0,1]$ find a trivializing neighbourhood say $U_{(x,t)} \times [t', t'']$ which contains the point (x,t). Hence $\{U_{x,t} \times [t', t'']\}_{t\in[0,1]}$ forms an open cover of $\{x\} \times [0,1]$ and let $\{U_{(x,t_i)} \times [t'_i, t''_i]\}_{i=1}^n$ be a finite subcover which exists since $\{x\} \times [0,1]$ is compact. Now, define $U_x := \bigcap_{i=1}^{n} U_{x,t_i}$. Now, using the fact that each $E|_{U_x\times[t',t'']}$ is trivializable and due to compactness of X there are at most finitely many bundles of the form $E|_{U_x\times[t',t'']}$, so using lemma 5.23 finitely many times, we get $E|_{U_x\times[0,1]}$ to be trivializable. Hence, we get get the required cover as $\{U_x\}_{x\in X}$. □

Lemma 5.25. *Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of a paracompact space X, then there is a countable open cover $\{V_k\}_{k\in\mathbb{N}}$ such that each V_k is a disjoint union of open sets each contained in some U_α, and there is a partition of unity $\{\phi_k\}_{k\in\mathbb{N}}$ with ϕ_k supported in V_k.*

Proof. Let $\{\phi_\alpha\}_{\alpha\in I}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in I}$. For each finite subset S of $\{\phi_\alpha\}_{\alpha\in I}$ define V_S to be the subset of X where all the ϕ_α's in S are strictly greater than all the ϕ_α's not in S . We claim that V_S is an open set. To prove this, let $x \in V_S$ be a point. Consider an open set U around U such that U intersects at most finitely many U_α's say, for $\alpha \in T$. Note that

$$x \in \left(U \cap \left(\bigcap_{\alpha\in S,\ \beta\in T-S} (\rho_\alpha - \rho_\beta)^{-1}(0,\infty)\right)\right) \subseteq V_S$$

since $V_S = \bigcup_{\alpha\in S,\ \beta\notin S} (\rho_\alpha - \rho_\beta)^{-1}(0,\infty)$. Clearly $U \cap \left(\bigcap_{\alpha\in S,\ \beta\in T-S} (\rho_\alpha - \rho_\beta)^{-1}(0,\infty)\right)$ is an open set, hence V_S is open. Also, V_S is contained in some U_α, namely, any U_α corresponding to $\phi_\alpha \in S$, since $\phi_\alpha \in S$ implies $\phi_\alpha > 0$ on V_S. Now, define V_k to be the union of all the open sets V_S such that S has k elements. In order to see that this is a disjoint union, let $x \in V_{S_1} \cap V_{S_2}$. Since S_1 and S_2 are two distinct sets of same finite cardinality, there exists elements $\phi_{\alpha_1} \in S_1 - S_2,\ \phi_{\alpha_2} \in S_2 - S_1$ such that $\alpha_1 \neq \alpha_2$, now, since $x \in V_{S_2}$, we can say that $\rho_2(x) > \rho_1(x)$, similarly, since $x \in V_{S_1}$ we can say that $\rho_1(x) > \rho_2(x)$ which is a contradiction. Now, the collection $\{V_k\}$ is

a cover of X since if $x \in X$ then $x \in V_S$ where, $S = \{\phi_\alpha | \phi_\alpha > 0\}$.
For the second statement, let $\{\phi_{k'}\}_{k' \in \mathbb{N}}$ be a partition of unity subordinate to the cover $\{V_k\}_{k \in \mathbb{N}}$, and define ϕ_k to be the sum of those $\phi_{k'}$'s which are supported in V_k but not in V_j for $j < k$. $\square$

Theorem 5.26. *(i) Let $\pi : E \to X$ be a vector bundle of rank n and $g : Y \to X$ be a continuous map. Then $(pr_2 =)\tilde{\pi} : g^*E \to X$ is a vector bundle of rank n over X and $\hat{g}(= pr_2)$ is a map of bundles.*

$$\begin{array}{ccc} g^*(E) & \xrightarrow{\hat{g}} & E \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{g} & X \end{array}$$

(ii) Let X be paracompact Hausdorff and let $g_0, g_1 : Y \to X$ be such that $\tilde{g} : Y \times [0,1] \to X$ is a one parameter family of maps, then $g_0^*(E)$ and $g_1^*(E)$ are isomorphic.

Proof. To prove (i), note that

$$\tilde{\pi}^{-1}(x) = \{(x, e) \in X \times E | \pi(e) = g(x)\} = \pi^{-1}(g(x)) \cong E_{g(x)}$$

Hence fiberwise vector space structure is clear. Now, for each $x \in X$ we have an open set $g^{-1}(U_{g(x)})$ say, v_x where, $U_{g(x)}$ is a trivializing neighbourhood of $g(x)$ in the vector bundle (E, π). Moreover, $g_{ij} \circ g : V_x \to GL_k(\mathbb{R})$ can act as the required transition functions since they satisfy the cocycle data, simply because,

$$(g_{ij} \circ g(x)) \cdot (g_{jk} \circ g(x)) = (g_{ij} \cdot g_{jk}) \circ g(x) = g_{ik} \circ g(x)$$

Now, to prove (ii), let $\{U_i\}_{i \in I}$ be trivializing neighbourhoods which cover X. Then, by lemma 5.25 we have a countable open cover say $\mathcal{V} = \{V_k\}_{k \in \mathbb{N}}$ and a partition of unity $\{\phi_k\}_{\in \mathbb{N}}$ subordinate to $\mathcal{V}$. Also, let $\rho_0 = 0$. Define

$$X_i = \text{ graph of } (\rho_0 + \rho_1 + \cdots + \rho_i) \text{ in } X \times [0,1] = \left\{ \left(x, \sum_{j=0}^{i} \rho_j(x) \right) \in X \times [0,1] \right\}$$

And $X_\infty := \left\{ \left(x, \sum_{j \in \mathbb{N}} \rho_j(x) \right) \in X \times [0,1] \right\} = \{(x, 1) \in X \times [0,1]\}$. Note that $X_i \cong X_j$ for all $i, j \in \mathbb{N} \cup \{0, \infty\}$. Now, define $E_i := E|_{X_i}$ for $i \in \mathbb{N} \cup \{0, \infty\}$. We aim to prove that $E_0 \cong E_\infty$ since $X_0 = X \times \{0\}$ and $X_\infty = X \times \{1\}$. To prove this, we will use the method of induction. So, let us construct a bundle map $F_i : E_i \to E_{i+1}$. First note that $\pi(e) = \left(x, \sum_{j=0}^{j=i} \rho_j(x) \right)$. If $pr_1 \circ \pi(e) \notin U_{i+1}$ then $e \mapsto e$, otherwise

$$e \xmapsto{\pi} \left(x, \sum_{j=0}^{i} \rho_j(x) \right) \xmapsto{\phi_{i+1}} \left(x, \sum_{j=0}^{i} \rho_j(x), v \right) \longmapsto \left(x, \sum_{j=0}^{i+1} \rho_j(x), v \right) \xmapsto{\phi_{i+1}^{-1}} F_i(e)$$

Note that F_i indeed is a homeomorphism, and fiberwise linear isomorphism, that is, F_i is a vector bundle isomorphism. Hence we are done.

$$\begin{array}{ccccccc}
E_0 & \xrightarrow[F_0]{\cong} & E_1 & \xrightarrow[F_1]{\cong} & E_2 & \xrightarrow[F_2]{\cong} & \cdots \\
\downarrow & & \downarrow{\scriptstyle Id} & & \downarrow & & \\
X_0 & \xrightarrow{\cong} & X_1 & \xrightarrow{\cong} & X_2 & \xrightarrow{\cong} & \cdots
\end{array}$$

□

Corollary 5.27. *Let (E, π) be a vector bundle of rank n over a paracompact, Hausdorff and contractible topological space X. Then (E, π) is isomorphic to a trivial bundle over X (that is, (E, π) is trivializable).*

Proof. Since X is contractible, there exists a one parameter family as follows: $\tilde{h} : X \times [0, 1] \to X$ such that $\tilde{h}|_{t=0} = Id_X$ and $\tilde{h}|_{t=1} = C_{x_0}$, where C_{x_0} is the constant function at a point $x_0 \in X$. Now, it can be seen from the method used in the proof of part (i) of the previous theorem that $h_0^*(E) \cong E$, and $h_1^*(E) \cong \varepsilon^n$ as vector bundles. Hence, we get the proof using part (ii) of the previous theorem. □

Lemma 5.28. *The Euclidean space $\mathbb{R}^n$ is paracompact for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ be fixed and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be a point. Now, define $V_{x,i} := B_1(x) \cap U_i$, for all $x \in \mathbb{R}^n, i \in I$, where $B_1(x)$ is an open ball of radius 1 and center x. We know $\overline{B_1(x)}$ is compact, due to the Heine Borel theorem. Hence, the open cover $\mathcal{U}$ of $\mathbb{R}^n$ which also covers $\overline{B_1(x)}$ has a finite subcover. Hence, we get the required refinement, namely $\{V_{x,i}\}_{x \in \mathbb{R}^n, i \in I}$ of $\{U_i\}_{i \in I}$ which is locally finite, since, for each $x \in \mathbb{R}^n$, there exists an open set $B_1(x)$ such that $\{(x, i) \in \mathbb{R}^n \times I | B_1(x) \cap V_{x,i} \neq \varnothing\}$ is a finite set. □

From corollary 5.27 we have vector bundles over paracompact, Hausdorff and contractible to be trivializable. The algebraic analogue of $\mathbb{R}^n$ is the affine n-space $\mathbb{k}^n$ (for a field $\mathbb{k}$). The ring of functions gets replaced here by polynomials, that is, $\mathbb{k}[x_1, \cdots, x_n]$. Algebraic vector bundles (that is, vector bundles with transition data as polynomials) correspond to projective modules over the polynomial ring in n variables. The question of whether every algebraic vector bundle is algebraically trivial (that is, the homotopy is algebraic) boils down to prove whether finitely generated projective modules are free. This was proved independently by Quillen and Suslin but in a more general context, that is, when $\mathbb{k}$ is not just a field, but a principal ideal domain. It is known as the Quillen-Suslin theorem.

Theorem 5.29 (Quillen-Suslin theorem)**.** *[11, 13] Let K be a principal ideal domain, $n \in \mathbb{N}$. Then every finitely generated projective module over the polynomial ring $K[x_1, \cdots, x_n]$ is free.*

Appendices

A Nakayama's lemma

We will refer [4] for this entire section.

Proposition A.1. *Let R be a commutative ring, M be a finitely generated R-module, let $\mathfrak{a}$ be an ideal of R, $\phi : M \to M$ be an R-module map such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form:*

$$\phi^n + a_1\phi^{n-1} + a_2\phi^{n-2} + \cdots + a_n = 0$$

where $a_i \in \mathfrak{a}$ for all $i \in \{1, 2, \cdots, n\}$.

Proof. Let $x_1, x_2, \cdots, x_n$ be generators of M. Since each $\phi(x_i) \in \mathfrak{a}M$, we can write $\phi(x_i) = \sum_{j=1}^{n} a_{ij} \cdot x_j$ for all $i \in \{1, 2, \cdots, n\}, a_{ij} \in \mathfrak{a}$ for all $i, j \in \{1, 2, \cdots, n\}$. After rearranging we get the following equation for each $i \in 1, 2, \cdots, n$.

$$\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij})x_j = 0$$

where δ_{ij} is the Kronecker delta. Now, consider the matrix with $(\delta_{ij}\phi - a_{ij})$ as ij^{th} entry and multiply both the sides by the adjoint matrix of $(\delta_{ij}\phi - a_{ij})$ on left. This gives a zero matrix. Hence, $det(\delta_{ij}\phi - a_{ij}) = 0$ since product of a matrix A with its adjoint matrix gives us a scalar matrix with the value of determinant of A on the diagonal entries. Hence using Cramer's rule [7] we get $det(\delta_{ij}\phi - a_{ij})$ to be a monic polynomial over ϕ with coefficients from $\mathfrak{a}$. This gives the required equation. □

Corollary A.2. *Let R be a commutative ring, M be a finitely generated R-module, let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}M = M$. Then there exists $x \in R$ such that $x \equiv 1$ (mod $\mathfrak{a}$) (that is, $(x - 1) \in \mathfrak{a}$) such that $xM = 0$.*

Proof. Take $\phi : M \to M$ to be the identity map, consider $x = 1 + a_1 + a_2 + \cdots + a_n$ which is obtained from proposition A.1, it also implies that $x = 0$, hence $xM = 0$. Moreover, $a_i \in \mathfrak{a}$ for all $i \in \{1, 2, \cdots, n\}$ hence $(x - 1) \in \mathfrak{a}$. □

Definition A.3. *Let R be a commutative ring. Define the Jacobson radical $\mathfrak{R}$ of ring R to be the ideal obtained after taking intersection of all maximal ideals in R. It is also denoted as $J(R)$.*

Proposition A.4. *Let $\mathfrak{R}$ be the Jacobson radical of a commutative ring R. An element $x \in \mathfrak{R}$ if and only if $(1 - xy)$ is a unit in R for all $y \in R$.*

Proof. (⇒) Suppose $(1 - xy)$ is not a unit, then it belongs the ideal generated by itself, and Zorn's lemma implies that this ideal lies in a maximal ideal, which in turn implies $(1 - xy)$ lies in $\mathfrak{R}$. But, $x \in \mathfrak{R}$ hence $xy \in \mathfrak{R}$ which implies $(1 - xy) + xy = 1 \in \mathfrak{R}$, which gives us a contraction.
(⇐) Suppose $x \notin \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ and x generate the full ideal, so we have $u + xy = 1$ for some $y \in R, u \in \mathfrak{m}$. Hence $1 - xy = u$ which implies $(1 - xy) \in \mathfrak{m}$, hence $(1 - xy)$ is not a unit which contradicts the hypothesis. □

Lemma A.5 (Nakayama's lemma)**.** *Let R be a commutative ring with unity, M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R contained in the Jacobson radical $\mathfrak{R}$ of R. Then $\mathfrak{a}M = M$ implies $M = 0$.*

Proof. By corollary A.2 we have $xM = 0$ for some $x \in R$ such that $x \equiv 1 \pmod{\mathfrak{R}}$. By proposition A.4, x is a unit in R since $(1 - x) \in \mathfrak{R}$ which implies $1 - (1 - x)y$ is unit for all $y \in R$. So, let $y = 1$ and we have the result. □

Lemma A.6. *Let R be a commutative ring, M be a finitely generated module over R, N be a submodule of M, and $M = N + \mathfrak{R}M$, then $M = N$.*

Proof. Consider the $A = M/N$. Note that A is also an R-module. Moreover, A is finitely generated, and we get $A = \mathfrak{R}A$ after taking the quotient modulo N of the equation $M = N + \mathfrak{R}M$. Hence, using lemma A.5 we get $A = 0$, that is, $M = N$. □

Proposition A.7. *Let R be a commutative ring, M be a finitely generated module over R and the elements $\{m_1 + \mathfrak{R}M, \cdots, m_n + \mathfrak{R}M\}$ generate $M/\mathfrak{R}M$ as an $R/\mathfrak{R}$-module, then $\{m_1, \cdots, m_n\}$ also generate M as an R-module.*

Proof. Define an R-module $N := \sum_{i=1}^{n} Rm_i$. Clearly, N is a submodule of M, hence it remains to prove that $M = N + \mathfrak{R}M$, for then applying lemma A.6 we get $M = N = \sum_{i=1}^{n} Rm_i$, that is, M is generated by $\{m_1, \cdots, m_n\}$. One sided inclusion, namely, $N + \mathfrak{R}M \subseteq M$ is trivial. For the reverse inclusion. Hence, let $m \in M$, let $\tilde{m}$ be its image under $M/\mathfrak{R}M$. Then we can write $\tilde{m} = \sum_{i=1}^{n} \lambda_i(m_i + \mathfrak{R}M)$, that is, $(m - \sum_{i=1}^{n} \lambda_i m_i) \in \mathfrak{R}M$, that is, $m \in N + \mathfrak{R}M$. This completes the proof. □

B Maximal ideals

Definition B.1 (Prime ideal). *(i) Let R be a commutative ring, then a proper ideal $\mathfrak{p}$ is called a prime ideal if for any elements a, b in R, $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ or both.*
(ii) Let R be a ring, then a proper ideal $\mathfrak{p}$ is called a prime ideal if for any ideals A, B in R, $AB \subseteq \mathfrak{p}$ implies either $A \subseteq \mathfrak{p}$ or $B \subseteq R$ or both.

Proposition B.2. *If R is a commutative ring, then the definitions (i) and (ii) are equivalent.*

Proof. Note that $\langle a\rangle\langle b\rangle = \langle ab\rangle$. ($\Leftarrow$) Let $ab \in \mathfrak{p}$, which implies $\langle ab\rangle \subseteq \mathfrak{p}$, that is $\langle a\rangle\langle b\rangle \subseteq \mathfrak{p}$ hence either $\langle a\rangle \subseteq \mathfrak{p}$ or $\langle b\rangle \subseteq \mathfrak{p}$ which implies that either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. ($\Rightarrow$) Let A, B be two ideals in R such that $AB \subseteq \mathfrak{p}$ but $A \not\subset \mathfrak{p}$ and $B \not\subset \mathfrak{p}$ which implies, there exists $a \in A - \mathfrak{p}, b \in B - \mathfrak{p}$, but $ab \in AB \subseteq \mathfrak{p}$, hence using definition (i) either of a or b belongs to $\mathfrak{p}$ which gives us a contradiction. □

Let A be a commutative ring with unity. Define a set Spec(A) := Set of all prime ideals of A. Let $E \subseteq A$, then define, $V(E) := \{\mathfrak{p} \in Spec(A) | E \subseteq \mathfrak{p}\}$. Now, we can define a topology on Spec(A) by calling a set $C \subseteq Spec(A)$ to be closed if and only if $C = V(E)$ for some ideal E in A. Let us check that this is indeed a topology.
(i) For $C = \varnothing$ consider $E = \langle 1\rangle$. Since prime ideals are proper ideals by definition, so $1 \notin \mathfrak{p}$ for any $\mathfrak{p} \in Spec(A)$.
(ii) For $C = Spec(A)$, consider $E = \langle 0\rangle$ since any ideal I contains 0 as $ax \in I$, for all $a \in A, x \in X$ and we can consider $a = 0$.
(iii) Let $\{V(E_i)\}_{i\in I}$ be closed sets, consider K to be the smallest ideal containing $\bigcup_{i\in I} E_i$ where, the definition of the smallest ideal containing a set S can be seen as the intersection of all the ideals containing the set S. This makes sense since intersection of ideals is also an ideal.
(iv) For closure under finite union, let us first consider $V(I_1), V(I_2)$ to be two closed sets. Consider $K = I_1 \cap I_2$. So, let us first prove that $V(I_1) \cup V(I_2) \subseteq V(I_1 \cap I_2)$. Let $\mathfrak{p} \in V(I_1)$ that is $\mathfrak{p} \in Spec(A)$ such that $I_1 \subseteq \mathfrak{p}$ which implies $I_1 \cap I_2 \subseteq I_1 \subseteq \mathfrak{p}$ that is $\mathfrak{p} \in V(I_1 \cap I_2)$, similarly $\mathfrak{p} \in V(I_2)$ implies $\mathfrak{p} \in V(I_1 \cap I_2)$. Now, to prove the reverse inclusion, let $\mathfrak{p} \in V(I_1 \cap I_2)$. This implies $\mathfrak{p} \in Spec(A)$ such that $I_1 I_2 \subseteq I_1 \cap I_2 \subseteq \mathfrak{p}$. Now, using primality of $\mathfrak{p}$, either $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$ or both which implies $\mathfrak{p} \in V(I_1)$ or $\mathfrak{p} \in V(I_2)$ or both respectively. We can extend the result using induction to get the proof. This topology is called Zariski topology.

Lemma B.3. *If M is a maximal ideal in a commutative ring R with unity then M is a prime ideal.*

Proof. Let us assume that M is not a prime ideal, then there exist $a, b \in \mathbb{R}$ such that $ab \in M$ but $a, b \notin M$. Consider the ideals $\langle a\rangle + M$ and $\langle b\rangle + M$. Using maximality $\langle a\rangle + M = R = \langle b\rangle + M$. Then, $ra + m = 1 = sb + n$ for some $r, s \in R, m, n \in M$. Now, $1 \cdot 1 = 1$ implies $(ra + m)(sb + n) = 1$ that is $rsab + ran + msb + mn = 1$. Now, since $ab, m, n \in M$ we have $1 \in M$ which gives us a contradiction since a maximal ideal is by definition a maximal ideal. $\square$

Now, for a ring A, we can put a subspace topology on $maxspec(A) := \{M \subseteq A \mid M$ is a maximal ideal$\}$ as a subspace of $Spec(A)$ which is equipped with the Zariski topology.

C Tychonoff spaces and Compactifications

We will refer [6, 24] for this entire section.

Definition C.1. *(i) A topological space X is said to be completely regular if for any given non-empty closed set $F \subseteq X$ and a point $x \in X - F$ there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0, f(F) = \{1\}$.*
(ii) A topological space X is said to be Tychonoff if it is completely regular and Hausdorff.

Remark C.2. *If X is completely regular space then for any given non-empty closed set $F \subseteq X$, a point $x \in X - F$, and $a, b \in \mathbb{R}$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = a, f(F) = \{b\}$.*

Lemma C.3. *If X is a normal space, then X is a Tychonoff space.*

Proof. Let X be a normal space. Let $F \subseteq X$ be a non-empty closed set, and $x \in X - F$ be a point. Since singleton sets are closed sets by the definition of normality, $\{x\}$ is a closed set and Urysohn's lemma implies that X is completely regular. Taking any two points $x, y \in X$ we can find disjoint open sets U_x, U_y containing x, y respectively, since singleton sets are closed sets by the definition of normality. Hence, X is Hausdorff, and so it is Tychonoff. □

Lemma C.4. *Let X be a paracompact Hausdorff space, then X is normal.*

Proof. Claim 1: If $x \in X$ and A is a closed set not containing x, then there exist disjoint open sets $U, V \subseteq X$ such that $x \in U, A \subseteq V$.
Proof: Since X is Hausdorff, for each $y \in A$ there exist disjoint open subsets U_y, V_y containing x, y respectively. Now, $\{V_y\}_{y \in Y}$ covers A and $\{V_y\}_{y \in A}, X - A$ forms an open cover of X. Thus, by paracompactness of X, there exists a locally finite open refinement, say $\mathcal{P}_1$. Throwing out all the subsets that do not intersect A, we will get a locally finite collection of open sets, each contained in some V_y, that cover A, call it $\mathcal{P}_2$. By locally finiteness of $\mathcal{P}_1$ there exists an open set W containing x such that only finitely many members of $\mathcal{P}_1$ intersect with W. Let $\tilde{A} = \{P \in \mathcal{P}_1 \mid P \cap W \neq \varnothing\}$. If $\tilde{A} \cap \mathcal{P}_2 = \varnothing$ then we are done, since we have open sets namely W containing x and $\bigcup_{U \in \mathcal{P}_2} U$ containing A, which are disjoint. Otherwise, using local finiteness of $\mathcal{P}_2$, we get an open set say W' which intersects with finitely many members of $\mathcal{P}_2$, say $B_1, B_2, \cdots, B_k$ and since these members are refinements of V_y's, we get a finite set say $T = \{y_i \in A \mid B_i \subseteq V_{y_i} \text{ for } i \in \{1, 2, \cdots, k\}\}$. Define, $U = W \cap \bigcap_{y \in T} U_y$ and $V = \bigcup_{Y \in \mathcal{P}_2} Y$, then $x \in U, A \subseteq V$ and $U \cap V = \varnothing$, and U, V are open sets.
Claim 2: If $A, B \subseteq X$ are disjoint closed subsets, then there exist $C, D \subseteq X$ open containing A, B respectively such that $C \cap D = \varnothing$.
Proof: From claim 1, for every $a \in A$, there exist open sets U_a containing a and V_a containing B such that $U_a \cap V_a = \varnothing$. Now, $\{U_a\}_{a \in A}$, $X - A$ forms an open cover of

X. Hence, it will have a locally finite open refinement. Throwing out from this any open subset not intersecting A, we still get a locally finite collection, say $\mathcal{P}$ of open subsets each contained in some U_a that cover A. Define, $C := \bigcup_{Y \in \mathcal{P}} Y$. Now, for each $b \in B$ we seek for an open subset D_b containing b such that $D_b \cap C = \varnothing$. Firstly, there exists an open subset W_b around b intersecting only finitely many members of $\mathcal{P}$. Note that, if we get a W_b which does not intersect with any member of $\mathcal{P}$ then we are already done. Let $B_1, B_2, \cdots, B_k$ be the elements of $\mathcal{P}$ contained in $U_{a_1}, U_{a_2}, \cdots, U_{a_k}$ respectively. Then, we can define $D_b = W_b \cap \bigcap_{i=1}^{k} V_{a_i}$. Now, $D := \bigcup_{b \in B} D_b$ will work as the required set. □

Remark C.5. *If a topological space is paracompact Hausdorff, then it is Tychonoff. Also, from lemma 4.14 we can say that, if a topological space is compact Hausdorff, then it is Tychonoff.*

Definition C.6. *(i) Let X, Y be topological spaces. A function $\phi : X \to Y$ is called an embedding if $\phi : X \to \phi(X)$ is a homeomorphism, where $\phi(X)$ has the subspace topology inherited from Y.*
(ii) Let X be a topological space. A compactification of X is a pair (K, h), where (a) K is compact Hausdorff space. (b) $h : X \to K$ is an embedding (c) $h(X)$ is a dense subset of K.
(iii) Let $(K_1, h_1), (K_2, h_2)$ be two compactifications of a same space X, then they are said to be equivalent, if there exists a homeomorphism $f : K_1 \to K_2$ such that $f \circ h_1 = h_2$.

Lemma C.7. *If X is a Hausdorff space and $A \subseteq X$ then A with the subspace topology is Hausdorff. If $\{X_i\}_{i \in I}$ is a collection of Hausdorff spaces, then $\prod_{i \in I} X_i$ is Hausdorff.*

Proof. Suppose $a, b \in A$ be distinct points. Since X is Hausdorff, there exists disjoint open sets $U, V \subseteq X$ containing a, b respectively. Then $U \cap A, V \cap A$ are disjoint open subsets in the subspace topology containing a, b respectively.
Now, suppose $x, y \in \prod_{i \in I} X_i$ be distinct. The points x, y are distinct, meaning there exists some $i \in I$ such that $x(i) \neq y(i)$ that is $x(i), y(i)$ are distinct points in X_i. So, since X_i's are Hausdorff, there exists disjoint open sets $U_i, V_i \subseteq X_i$ containing x_i, y_i respectively. Let, $U := \pi_i^{-1}(U_i), V := \pi_i^{-1}(V_i)$, where π_i is the projection map from $\prod_{i \in I} X_i$ to X_i. The topology on $\prod_{i \in I} X_i$ is defined in such way that, it is the weakest topology which makes π_i continuous, for all $i \in I$. Hence, U, V are open sets. Hence, we get disjoint open sets U, V containing x, y respectively. □

Definition C.8. *Let $\{X_i\}_{i \in I}, X$ be topological spaces, $f : X \to X_i$ be continuous functions.*
(i) The initial topology induced by $\{f_i\}_{i \in I}$ is the weakest topology on X such that f_i's are continuous, for all $i \in I$. That is, a subbase is collection of sets of the form $f_i^{-1}(U_{i_j})$ where $i \in I, U_{i_j} \subseteq X_i$ is open.

(ii) The evaluation map is the function $e : X \to \prod_{i \in I} X_i$ *defined by* $(\pi_i \circ e)(x) = f_i(x)$ *for all* $i \in I$.

Lemma C.9. *If* X *is completely regular and* $A \subseteq X$*, then* A *with the subspace topology is completely regular. If* $\{X_i\}_{i \in I}$ *is a collection of completely regular spaces, then* $\prod_{i \in I} X_i$ *is completely regular.*

Proof. Suppose F is closed in A, and $x \in A - F$, then there is a closed set $G \subseteq X$ with $F = G \cap A$. Then $x \notin G$ (since, if $x \in G$, then $x \in G \cap A = F$, which is a contradiction). So, there is a function $f : X \to [0, 1]$ such that $f(x) = 0, f(G) = \{1\}$. So, we can consider $f|_A$ which is continuous since f is continuous and A has the subspace topology. Now suppose that F is a closed set of $\prod_{i \in I} X_i$ and $x \in X - F$. A base for the product topology consists of intersections of finitely many sets of the form $\pi_i^{-1}(U_{i_j})$, where $i \in I$, $U_{i_j} \subseteq X_i$ are open. Now, since $X - F$ is an open set containing x, there is a finite subset K such that $x \in \bigcap_{i_j \in K} \pi_i^{-1}(U_{i_j}) \subseteq X - F$. For each $i_j \in K$, $X_i - U_{i_j}$ is closed, and $x(i) \in U_{i_j}$ and since X_i is completly regular, there exists a continuous function $f_i : X_i \to [0, 1]$ such that $f_{i_j}(x(i)) = 0, f_i(X_i - U_{i_j}) = \{1\}$. Define $g : X \to [0, 1]$ by $g := \max_{i_j \in K} (f_{i_j} \circ \pi_j)(y)$ for all $y \in X$. Since K is a finite set, the maximum function makes sense, moreover g is continuous since maximum of finitely many continuous functions is also a continuous function. As we can write $\max(f, g) = \dfrac{f + g + |f - g|}{2}$ and $|x|$ is a continuous function , and so is the composition and addition and subtraction of continuous functions. So, we get a function $g : X \to [0, 1]$ such that $g(x) = 0$, since $(f_{i_j} \circ \pi_i)(x) = 0$ for all $i_j \in K$ and $F \subseteq X - \bigcap_{i_j \in K} \pi_i^{-1}(U_{i_j})$ that is $F \subseteq X \cap \left(\bigcap_{i_j \in K} \pi_i^{-1}(U_{i_j}) \right)^C = \bigcup_{i_j \in K} \pi_i^{-1}(U_{i_j})$. So, if $y \in F$, there exists some $i_j \in K$ such that $\pi_i(y) \in X_i - U_{i_j}$ which implies $(f_j \circ \pi_j)(y) = 1$, which implies $g(F) = \{1\}$ hence the proof is complete. □

Remark C.10. *Let* X *be a Tychonoff space and* $A \subseteq X$*, then* A *with the subspace topology is Tychonoff. If* $\{X_i\}_{i \in I}$ *is a collection of Tychonoff spaces, then* $\prod_{i \in I} X_i$ *is Tychonoff.*

Definition C.11. *A topological space* X *is said to be locally compact if for each* $x \in X$ *there exists an open set* U_x *containing* x *such that* $\overline{U_x}$ *is compact.*

Example C.12. *A compact space* X *is locally compact since* $U_x = X$ *works for all* $x \in X$.

Theorem C.13. *Let* X *be a locally compact (non-compact) Hausdorff space. Then there exists unique compactification* $(\hat{X}, h)$ *such that* $|\hat{X} - h(X)| = 1$*. This is called one-point compactification of* X*. It is also called Alexandroff extension of* X*.*

Proof. Let us denote $\hat{X} = X \cup \{\infty\}$ where ∞ is a point which does not belong to X. We define a topology on $\hat{X}$ by calling a set U to be open if (i) U is open in X. or (ii) $U = (X - C) \cup \{\infty\}$ for some compact set $C \subseteq X$. Now, let us verify that this indeed forms a topology. Since $\varnothing \subseteq X$ is open in X hence $\varnothing$ is an open set in $\hat{X}$. Now, since $\varnothing$ is a compact set in X, hence $(X - \varnothing) \cup \{\infty\} = \hat{X}$ is an open set in $\hat{X}$. Now, let $\{U_i\}_{i\in I}$ be a collection of open sets. Let $J \subseteq I$ be the set consisting of the indices of open sets of type (i), $K := I - J$. Let $\mathcal{U}_J := \bigcup_{j\in J} U_j$. Since union of open sets is open, $\mathcal{U}$ is open in X, and so is in $\hat{X}$. Let us assume that, for $i \in K$, $U_i = (X - C_i) \cup \{\infty\}$ where $C_i \subseteq X_i$ is compact. Observe that

$$X - U_i = X \cap U_i^C = X \cap (X - C_i^C \cap \{\infty\}^C) = \cap(X \cap C_i^C)^C \cap X$$

and we know that

$$X \cap (X \cap C_i^C)^C \cap X = X \cap (X^C \cup C_i) = \varnothing \cup C_i = C_i.$$

Denote $\mathcal{U}_K := \bigcup_{i\in K} U_i$, we have $X - \mathcal{U}_K = \bigcap_{i\in K}(X - U_i)$ which implies $X - \mathcal{U}_K = \bigcap_{i\in K} C_i =: C_K$. As X is Hausdorff and C_i's are compact, we have C_i's to be closed in X. So, C_K is also closed, moreover it is compact since it is closed subset of a compact set say C_{i_0} for some $i_0 \in K$. Note that, if there does not exist any such i_0, then we are already done. Now, we have $\mathcal{U}_I = \mathcal{J} \cup \mathcal{K}$ which implies $X - \mathcal{U}_I = (X - \mathcal{U}_J) \cap (X - \mathcal{U}_K)$ and we have shown that these two sets are closed in X, hence their intersection is also closed in X. Hence, $(X - \mathcal{U}_I)$ is compact, since it is closed subset of a compact set $(X - \mathcal{U}_K)$. Hence, $\mathcal{U}_I$ is an open set of type (ii) in $\hat{X}$. Now to show that finite intersection of open sets is open. For this, we will do following cases:

Case I: U_1, U_2 be open sets, both of type (i) then $U_1 \cap U_2$ is open in X and so is in $\hat{X}$.

Case II: U_1, U_2 be open sets, both of type (ii), then $(X - U_1) \cup (X - U_2)$ is compact since union of finitely many compact sets is compact. This implies $X - (U_1 \cap U_2) = (X - U_1) \cup (X - U_2)$ is compact, hence $U_1 \cap U_2$ is open, by type (ii).

Case III: Without loss of generality, let us assume that U_1 is an open set of type (i) and U_2 of type (ii). Consider $(X - U_1) \cup (X - U_2)$. This is closed in X, since $X - U_2$ is compact and hence it is closed as X is Hausdorff. Also, since $\infty \notin U_1$ hence $\infty \notin U_1 \cap U_2$. Hence, it is enough to show that $X - (U_1 \cap U_2)$ is closed in X which is true since $X - (U_1 \cap U_2) = (X - U_1) \cup (X - U_2)$. Hence, the structure defined on $\hat{X}$ is indeed a topology.

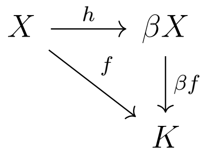
Now, let us show that $\hat{X}$ is compact. Consider an open cover of $\hat{X}$. One of the sets contain the point ∞, which is of the form $(X - C) \cup \{\infty\}$ for some compact set $C \subseteq X$. Then we need only finitely many open sets to cover C which we get from the compactness of C. To see that $\hat{X}$ is Hausdorff, let $x, y \in \hat{X}$ be distinct. If $x, y \in X$, then we are done, since X is Hausdorff and open sets of X are open set of $\hat{X}$. If $y = \infty$. Since X is locally compact, there exists an open set U_x containing x such that $\overline{U_x}$ is compact. So, $(X - \overline{U_x}) \cup \{\infty\}$ and U_x are disjoint open sets containing

∞ and x respectively. Consider the map $h : X \to \hat{X}$ to be the identity map. So, it remains to show that X is dense in $\hat{X}$. Let $U_C = (X - C) \cup \{\infty\}$ be a neighbourhood of ∞. Since X is not compact, that is $X \neq C$ so, there must be some point $x \in X - C$. That is $\infty \in \overline{X}$.
To prove the uniqueness part, let Y be a compact Hausdorff space such that $Y - X = \{\infty\}$ and $\overline{X} = Y$. Where Y has a topology such that we get back the topology on X if we restrict the topology of Y to X. Let $\tau_{\hat{X}}$ denote the topology on the one-point compactification according to the proof of existence, and let τ_Y denote the topology on space Y.Let us first prove that $\tau_Y \subseteq \tau_{\hat{X}}$. Let U be an open set in Y. If $\infty \notin U$, then U is open in X, hence U is open in $\tau_{\hat{X}}$ since it is open set of the type (i). If $\infty \in U$ then let $V := U \cap X$, then if $V = \varnothing$ which implies $U = \{\infty\}$ which in turn implies $X = Y - \{\infty\}$ is a closed set, hence it is compact since it is closed subset of a compact set Y. So, there is an open set $(X - C) \cup \{\infty\}$ where C can be replaced by X which implies $\{\infty\}$ is open in $\hat{X}$. Now, if $V \neq \varnothing$, then V being open in X, $X - V$ is closed in X. Moreover, $X - V = Y - U$ is also closed in Y since U is open in Y. Hence, $Y - U$ is compact and so is $X - V$ that is every open neighbourhood of ∞ in Y is the complement of a compact subset of X. Thus $\tau_Y \subseteq \tau_{\hat{X}}$. Now, if $U \subseteq X$ is open, then clearly $U \in \tau_Y$. So, let $K \subseteq X$ compact and let $V := X - K \subseteq X$. Since X is Hausdorff, K is closed since it is compact, hence V is open. Note that $Y - K = V \cup \{\infty\}$. If $Y - K = V \cup \{\infty\} \in \tau_Y$ then we are done. But if $V \cup \{\infty\} \notin \tau_Y$ which means, there exists a point $x \in V \cup \{\infty\}$ for which there does exists any open set which is contained in $V \cup \{\infty\}$. If $x \in V$, then V can act the open set which is contained in $V \cup \{\infty\}$. So, $x = \infty$ is the only point for which every open set has non-empty intersection with $(Y - K)^C (= K)$. In other words $\infty \in \overline{K}$. But this is not possible since X is Hausdorff, so K is closed because it is it is compact. So, $K = \overline{K}$, but $\infty \notin K$ and $\infty \in \overline{K}$. This gives us a contradiction and this completes the proof. ☐

Lemma C.14. *Let X be a topological space, $A \subseteq X$, Y be a Hausdorff space, and let $f, g : X \to Y$ be two continuous function such that $f \equiv g$ on A then $f \equiv g$ on $\overline{A}$.*

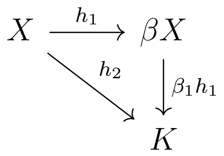
Proof. Let $x \in \overline{A} - A$ such that $f(x) \neq g(x)$. Then, since Y is Hausdorff, there exists disjoint open sets U_f, U_g containing $f(x), g(x)$ respectively, Now, consider $V_f := f^{-1}(U_f), V_g := g^{-1}(U_g)$ which are open sets since f, g are continuous. Also, both V_f and V_g contain x. Now, since $x \in \overline{A} - A$, for any open set U around x, there exists a point $p \in (U - \{x\} \cap A)$ that is $(U - \{x\} \cap A) \neq \varnothing$. Consider one such point p corresponding to the non-empty open set $V_f \cap V_g$, then $f(p) \in U_f$ and $g(p) \in U_g$ and $f(p) = g(p)$, which makes $U_f \cap U_g \neq \varnothing$. This is a contradiction. ☐

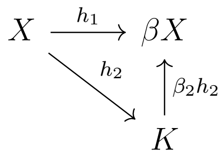
Definition C.15. *Let X be a topological space. Then a compactification denoted as $(\beta X, h)$ is called a Stone-Čech compactification of X if, given any continuous map $f : X \to K$, where K is compact and Hausdorff, there exists unique map $\beta f : \beta X \to K$ such that the following diagram commutes.*



Proposition C.16. *Let X be topological space, then its Stone-Čech compactification is unique, if it exists.*

Proof. On contrary, let us assume that $(\beta_1 X, h_1)$ and $(\beta_2 X, h_2)$ are two compactifications which satisfies the property mention in the definition of Stone-Čech compactification (it is also called a universal property). Hence, we can have the following diagrams, which commute:





If $y \in \beta_1 X$ such that $y \in h_1(X)$, then we get that $(\beta_1 h_1) \circ (\beta_2 h_2) = Id$ on $h_1(X)$ and from lemma C.14 we can say that, $(\beta_1 h_1) \circ (\beta_2 h_2) = Id$ on $\overline{h_1(X)} = \beta_1(X)$. Similarly, $(\beta_2 h_2) \circ (\beta_1 h_1) = Id$ on $\beta_2 X$, which gives us the required homeomorphism. □

Lemma C.17. *Let X be a topological space. If X has a compactification, then X is Tychonoff.*

Proof. Let (K, h) be a compactification of space X. Since, K is compact Hausdorff, K is normal by lemma 4.14 and using Urysohn's lemma we get K to be completely regular, and K is Hausdorff as well, so K is Tychonoff hence due to the remark C.10, $h(X)$ with the subspace topology is Tychonoff and since X and $h(X)$ are homeomorphic, X is Tychonoff. □

Remark C.18. *If X is a locally compact Hausdorff space, then X is Tychonoff.*

Since every topological space which has compactification is Tychonoff, so one can naturally ask if the converse is true? That is, if X is Tychonoff, then does there exist a compactification of X? The answer is yes! We will explicitly construct the compactification through following series of propositions.

Proposition C.19. *Let X, $\{X_i\}_{i\in I}$ be topological spaces, and $f_i : X \to X_i$ be continuous functions. Then the evaluation map $e : X \to \prod_{i\in I} X_i$ is an embedding if and only if (i) X has the initial topology induced by the family $\{f_i\}_{i\in I}$ and (ii) the family $\{f_i\}_{i\in I}$ separates points in X.*

Proof. Write $P = \prod_{i\in I} X_i$ and let $p_i : e(X) \to X_i$ be the restriction of $\pi_i : P \to X_i$ to $e(X)$. A subbase for $e(X)$ with the subspace topology inherited from P consists of

the sets of the form $\pi_i^{-1}(U_{i_j}) \cap e(X), i \in I, U_{i_j} \subseteq X_i$ open. But $\pi_i^{-1}(U_{i_j}) \cap e(X) = p_i^{-1}(U_{i_j})$ and the collection of this form is a subbase for $e(X)$ with the initial topology induced by the family $\{p_i\}_{i \in I}$, these topologies are equal.
($\Rightarrow$) Assume that $e : X \to e(X)$ is a homeomorphism and since $f_i = \pi_i \circ e = p_i \circ e$, $e(X)$ having the initial topology induced by $\{p_i\}_{i \in I}$ implies that X has initial topology induced by $\{f_i\}_{i \in I}$. If $x, y \in X$ be distinct points, then there exists some $i \in I$ such that $p_i(e(x)) \neq p_i(e(y))$ since e is a homeomorphism and so it is injective, so $e(x), e(y)$ are distinct and $\{p_i\}_{i \in I}$ separates points. This implies $f_i(x) \neq f_i(y)$, which shows that $\{f_i\}_{i \in I}$ separates points in X.
($\Leftarrow$) It suffices to prove that $e : X \to P$ is one-to-one, continuous, and that $e : X \to e(X)$ is an open map. If $x, y \in X$ are distinct points, then because $\{f_i\}_{i \in I}$ separates points in X, there exists some $i \in I$ such that $f_i(x) \neq f_i(y)$ which implies $e(x) \neq e(y)$, showing that e is one-to-one. For each $i \in I$, f_i is continuous and $f_i = \pi_i \circ e$. The fact that this is true for all $i \in I$ implies $e : X \to P$ is continuous, since product topology is the initial topology induced by the family of projection maps, that is a map to the product is continuous if and only if its composition with each projection map is continuous.
A subbase for the topology of X consists $V = f_i^{-1}(U_{i_j}), i \in I, U_{i_j} \subseteq X_i$ open. As $f_i = p_i \circ e$, we can write $V = (p_i \circ e)^{-1}(U_{i_j}) = e^{-1}(p^{-1}(U_{i_j}))$, and since e is one-to-one, applying e on both the sides, we get $e(V) = p_i(U_{i_j})$ which is open in $e(X)$ since p_i's are continuous functions. $\square$

Definition C.20. *Let $X, \{X_i\}_{i \in I}$ be topological spaces. For each $i \in I$, $f_i : X \to X_i$ be continuous functions. Then we say $\{f_i\}_{i \in I}$ separates points from closed sets if whenever $F \subseteq X$ is closed and $x \in X - F$, there is some $i \in I$ such that $f_i(x) \notin \overline{f_i(F)}$.*

Proposition C.21. *Let $X, \{X_i\}_{i \in I}$ be topological spaces. For each $i \in I$, $f_i : X \to X_i$ be continuous functions. The family $\{f_i\}_{i \in I}$ separates points from closed sets if and only if the collection of sets of the form $f_i^{-1}(U_{i_j}), i \in I, U_{i_j} \subseteq X_i$ open, is a base for the topology on X.*

Proof. ($\Rightarrow$) Let $x \in X$ and U be an open neighbourhood of x. Then $F = X - U$ is closed, so there is some $i \in I$ such that $f_i(x) \notin \overline{f_i(F)}$. Consider $U_i := X_i - \overline{f_i(F)}$ which is open in X_i because $\overline{f_i(F)}$ is closed. Hence, $f_i^{-1}(U_i)$ is also open X. On the other hand $f_i(x_i) \in U_i$ which implies $x_i \in f_i^{-1}(U_i)$. Also, if $y \in f_i^{-1}(U_i)$ that is $f_i(y) \in U_i$ which implies $y \notin F$. Because, if $y \in F$, then $f_i(y) \in F$ that is $f_i(y) \in U_i \cap f_i(F)$ but we have $U_i \cap f_i(F) = \varnothing$ since $U_i = X_i - \overline{f_i(F)}$. So, this implies $y \in U$, which in turn implies $f_i^{-1}(U_i) \subseteq U$.
($\Leftarrow$) Let $F \subseteq X$ be closed, and $x \in X - F$ be a point. Because $X - F$ is a neighbourhood of x, there exists some $i \in I$ and $U_{i_j} \subseteq X_i$ open such that $x \in f_i^{-1}(U_{i_j}) \subseteq X - F$. So, $f_i(x) \in U_{i_j}$. Now suppose there is some $y \in F$ such that $f_i(y) \in U_{i_j}$, this implies $y \in f_i^{-1}(U_{i_j}) \subseteq X - F$ which contradicts $y \in F$. Therefore $U_{i_j} \cap f_i(F) = \varnothing$ and hence $X_i - U_{i_j}$ is a closed set that contains $f_i(F)$, which tells us that $\overline{f_i(F)} \subseteq X_i - U_i$ since $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. So, we get $\overline{f_i(F)} \cap U_{i_j} = \varnothing$, but $f_i(x) \in U_{i_j}$ hence, $f_i(x) \notin \overline{f_i(F)}$, which completes the proof. $\square$

Proposition C.22. *If X is a T_1 space, $\{X_i\}_{i\in I}$ are topological spaces, $f_i : X \to X_i$ are continuous functions and $\{f_i\}_{i\in I}$ separates points from closed sets in X, then the evaluation map $e : X \to \prod_{i\in I} X_i$ is an embedding.*

Proof. By proposition C.21 there exists a base for the topology of X of the form $f_i^{-1}(U_{i_j}), i \in I, U_{i_j} \subseteq X_i$ open. Since a base is also a subbase, topology generated by this subbase is the initial topology for the family of functions $\{f_i\}_{i\in I}$. Because X is T_1, singletons are closed and therefore, the fact that $\{f_i\}_{i\in I}$ separates points point from closed sets implies that it separates points in X. Therefore, we can apply proposition C.19, which tells us that the evaluation map is an embedding. □

Proposition C.23. *Let X be a topological space. Then X is completely regular if and only if X has the initial topology induced by $C_b(X) := \{f \in C(X) \mid f \text{ is bounded }\}$.*

Proof. ($\Rightarrow$) If $F \subseteq X$ is closed and $x \in X - F$, then there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0, f(F) = \{1\}$. Then, $f \in C_b(X)$ since its codomain itself is bounded. Moreover, $f(x) = 0 \notin \{1\} = \overline{\{1\}} = \overline{f(F)}$, which shows that $C_b(X)$ separates points from closed sets in X. So, we are done by applying proposition C.21 since, a base is also a subbase.
($\Leftarrow$) Now, suppose $F \subseteq X$ is closed and $x \in X - F$. A subbase for initial topology induced by $C_b(X)$ consists of sets of the form $f^{-1}(V), f \in C_b(X), V \subseteq X$ an open ray in $\mathbb{R}$, because the set of open rays are a subbase for the topology of $\mathbb{R}$. Since $X - F$ is an open neighbourhood of x, there exists a finite set $J \subseteq C_b(X)$ and open rays V_f in $\mathbb{R}$ for each $f \in J$ such that $x \in \bigcap_{f\in J} f^{-1}(V_f) \subseteq X - F$. If some V_j is of the form $(-\infty, a_f)$, then with $g = -f$ we have $f^{-1}(-\infty, a_f) = g^{-1}(-a_f, \infty)$. We therefore suppose $V_f = (a_f, \infty)$ for each $f \in J$. Now, for each $f \in J$ define $g_f(x) := \max\{f(x) - a_f, 0\}$ which is a continuous and non-negative function satisfying $f^{-1}(a_f, \infty) = g_f^{-1}(0, \infty)$. Define $g := \prod_{f\in J} g_f$. Note that J is finite, hence g makes sense and it is continuous and positive. If $y \in g^{-1}(0, \infty)$, then $y \in \bigcap_{f\in J} g_f^{-1}(0, \infty) \subseteq X - F$ so, $g^{-1}(0, \infty) \subseteq X - F$. But g is non-negative, so $g(X - (X - F)) = g(F) = \{0\}$, which implies X is completely regular. □

Definition C.24. *A cube is a topological space that is homeomorphic to a product of compact intervals in $\mathbb{R}$.*

A product is homeomorphic to the same product without singleton factor. For example, $\mathbb{R} \times \mathbb{R} \times \{3\} \cong \mathbb{R} \times \mathbb{R}$ and a product of non-singleton compact intervals with index set I is homeomorphic to $[0, 1]^I$.

Lemma C.25. *If a topological space X is Hausdorff, then X is T_1.*

Proof. Let $x \in X$ be a point. Let us show that $\{x\}^C$ is an open set. So, consider a point $y \neq x$. Since, X is Hausdorff, there exists disjoint open sets U_x, U_y containing x, y respectively. In particular $U_y \subseteq X - \{x\}$ hence, $\{x\}^C$ is open. □

Proposition C.26. *A topological space X is Tychonoff if and only if X is homeomorphic to a subspace of a cube.*

Proof. ($\Leftarrow$) Suppose I is a set and X is homeomorphic to a subspace Y of $[0,1]^I$. Since $[0,1]$ is Tychonoff, so is the product $[0,1]^I$ and so is the subspace Y. Hence X is Tychonofff.
($\Rightarrow$) By proposition C.23, X has the initial topology induced by $C_b(X)$. For each $f \in C_b(X)$, let $I_f := [-\|f\|_\infty, \|f\|_\infty]$ and $f : X \to I_f$ is continuous. Now, since X is Tychonoff, it is Hausdorff hence by lemma C.25 it is T_1 and since $\{f\}_{f \in C_b(X)}$ separates points from closed sets, since X is completely regular. Hence, we can now apply proposition C.22 which tells us that $e : X \to \prod_{f \in C_b(X)} I_f$ is an embedding. □

Now, note that, for each $f \in C_b(X)$, the interval $[-\|f\|_\infty, \|f\|_\infty]$ is compact amd Hausdorff. Also, if $f = 0$, then $I_f = \{0\}$ which is is indeed compact and vacuously Hausdorff. So, the product $\prod_{f \in C_b(X)} I_f$ is compact and Hausdorff since product of copmact sets is compact by Tychonoff's theorem. Since closed subset of a compact set is compact, we define βX to be $\overline{e(X)} \subseteq \prod_{f \in C_b(X)} I_f$. We claim that the compactification $(\beta X, e)$ is the Stone-Čech compactification. So, let us check that it satisfies the universal property.

Lemma C.27. *If X, Y are topological spaces and $f : X \to Y$ is a continuous map, then for any subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$*

Proof. We have, $A \subseteq f^{-1}(\overline{f(A)})$ and pre-image of closed set is closed under a continuous map, so $f^{-1}(\overline{f(A)})$ will be closed. Hence, $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$ since $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. Hence, $f(\overline{A}) \subseteq \overline{f(A)}$. □

Theorem C.28. *If X is a Tychonoff space, K is a compact Hausdorff space, and $\phi : X \to K$ is continuous, then there exists a unique cotinuous fucntion $\Phi : \beta X \to K$ such that $\phi = \Phi \circ e$, that is, $(\beta X, e)$ is the Stone-Čech compactification of X.*

Proof. The space K is a Tychonoff space since a compact Hausdorff space is normal and a normal space is Tychonoff. So, the evaluation map $e_K : K \to \prod_{g \in C_b(K)} I_g$ is an embedding. Write $F = \prod_{f \in C_b(X)} I_f, G = \prod_{g \in C_b(K)} I_g$, and let $p_f : F \to I_f, q_g : G \to I_g$ be the projection maps. We define $H : F \to G$ for $t \in F$ by $(q_g \circ H)(t) = p_{g \circ \phi}(t)$ for each $g \in C_b(K)$. The map $q_g \circ H : F \to I_{g\circ\phi}$ is continuous for all $g \in G$, hence H is continuous. For each $x \in X$, we have

$$(q_g \circ H \circ e)(x) = p_{g\circ\phi}(e(x)) = (p_{g\circ\phi} \circ e)(x) = (g \circ \phi)(x) = (q_g \circ e_K)(\phi(x)) = (q_g \circ e_K \circ \phi)(x)$$

so, we have the following result.

$$q_g \circ H \circ e \equiv q_g \circ e_K \circ \phi \text{ for all } g \in C_b(K). \tag{15}$$

This is equivalent to the fact that $H \circ e = e_K \circ \phi$. Now, because K is compact and e_K is continuous, hence $e_K(K)$ is compact, hence closed, since it is a compact subset of a Hausedorff space G. From 15, we know $H(e(X)) \subseteq e_K(K)$ since $\phi(X) \subseteq K$, thus $\overline{H(e(X))} \subseteq \overline{e_K(K)} = e_K(K)$ which implies $\overline{H(e(X))} \subseteq e_K(K)$. On the other hand, because βX is compact and H is continuous, $H(\beta X)$ is compact and hence it is closed subset of G, since G is Hausdorff. Using lemma C.27 and the fact that $f(A) \subseteq f(\overline{A})$ implying $\overline{f(A)} \subseteq \overline{f(\overline{A})}$, where f can be replaced by H and A by $e(X)$, we get $\overline{H(e(X))} = \overline{H(\beta X)} = H(\beta X)$ since $H(\beta X)$ is closed. Hence we have

$$H(\beta X) \subseteq e_K(K). \tag{16}$$

Hence, letting h to be the restriction of H to βX and define $\Phi : \beta X \to K$ by $\Phi = e_K^{-1} \circ h$ which makes sense due to 16 and the fact that $e_K : K \to e_K(K)$ is a homeomorphism. Now, using 15,

$$(\Phi \circ e)(x) = (e_K^{-1} \circ h \circ e)(x) = (e_K^{-1} \circ H \circ e)(x) = \phi(x)$$

showing that $\Phi \circ e = \phi$. Now, for the uniqueness part, let $\Psi : \beta X \to K$ be a continuous map satisfying $f = \Psi \circ e$. Let, $y \in e(X)$, then there is some $x \in X$ such that $y = e(x)$ which implies $f(x) = (\Psi \circ e)(x) = \Psi(y)$, also, $f(x) = (\Phi \circ e)(x) = \Phi(y)$ which implies $\Phi(y) = \Psi(y)$ for all $y \in e(X)$. Now, since Φ, Ψ are continuous, the codomain K is Hausdorff and are equal on $e(X)$, so using lemma C.14 they are equal on $\overline{e(X)} = \beta X$. This completes the proof. $\square$

From the remark C.18 we saw that every locally compact Hausdorff space is Tychonoff. So, is it the case that the one-point compactification of a locally compact Hausdorff space is same as its Stone-Čech compactification? The answer is that, it is not always same. For example $X = \mathbb{N}$ with subspace topology inherited from $\mathbb{R}$. $\mathbb{N}$ is locally compact, since every point $n \in \mathbb{N}$ has an open set $U_n = \{n\}$. Moreover, $U_n = \overline{U_n}$ and $\overline{U_n}$ is compact since it is a finite set. It is known that $|\beta\mathbb{N}| = 2^C$, where $C = |\mathbb{R}|$ [18, 19], and one-point compactification of $\mathbb{N}$ has cardinality of order less than that of C. But, there are spaces like $[0, \omega_1)$ with the order topology where the Stone-Čech compactification is same as the one-point compactification [12]. From remark C.5 and theorem C.28 we can say that a paracompact Hausdorff space admits a Stone-Čech compactification. Now, let us see, if paracompact Hausdorff spaces are locally compact Hausdorff so as to talk about the relation between their one-point compactification (if exists) with their Stone-Čech compactification.

But we find that there are spaces which are paracompact Hausdorff, but not locally compact Hausdorff. For example $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$. So, let us ask the question again that, if we enforce the condition of paracompactness with locally compact Hausdorff spaces, can we get one-point compactification same as Stone-Čech compactification? Again, the answer is no. An example for this is $\mathbb{R}$ with the usual topology. This is again due to a cardinality argument that $|\beta\mathbb{R}| = |\beta\mathbb{N}| = 2^C$ [18, 19], whereas the order of cardinality of one-point compactification of $\mathbb{R}$ is C. One can look at the one-point compactifications as the smallest compactifications, whereas Stone-Čech compactification as the biggest. So, it is quite unlikely that they coincide.

Bibliography

[1] Existence of maximal ideals. https://equatorialmaths.wordpress.com/2009/03/11/existence-of-maximal-ideals, 2009.

[2] Colimits of paracompact hausdorff spaces. https://ncatlab.org/nlab/show/colimits+of+paracompact+Hausdorff+spaces, 2017.

[3] Micheal Atiyah. *K-theory*. W. A. Benjamin Inc., 1967.

[4] Micheal Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. University of Oxford, 1994.

[5] Stefan Banach. Théorie des opérations linéaires. *Glasgow Mathematics Journal-Warszawa: Instytut Matematyczny Polskiej Akademii Nauk*, 1932.

[6] Jordan Bell. The Stone-Čech compactification of tychonoff spaces. *University of Toronto*, 2014.

[7] Gabriel Cramer. Cramer's rule. https://en.wikipedia.org/wiki/Cramer%27s_rule, 1750.

[8] Eisenberg, Murray, Guy, and Robert. A proof of the hairy ball theorem. *The American Mathematical Monthly*, 1979.

[9] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[10] Daniel Hudson. A bridge between algebra and topology: Swan's theorem.

[11] Serge Lang. *Algebra*. Springer, 2002.

[12] Dan Ma. The first uncountable ordinal. https://dantopology.wordpress.com/2009/10/11/the-first-uncountable-ordinal/, 2009.

[13] Akhil Mathew. The Quillen-Suslin theorem. https://amathew.wordpress.com/2012/01/16/the-quillen-suslin-theorem/, 2012.

[14] John Milnor. *Topology from the Differentiable Viewpoint*. Princeton University Press, 1997.

[15] John Milnor and James Stasheff. *Characteristic Classes*. Princeton University Press, University of Tokyo Press, 1974.

[16] Jason Polak. Projective modules over local rings are free. http://blog.jpolak.org/?p=363, 2012.

[17] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book company, 1987.

[18] Brian Scott. Stone Čech compactification of $\mathbb{N}, \mathbb{Q}, \mathbb{R}$. https://math.stackexchange.com/questions/96051/stone-%C4%8Cech-compactification-of-mathbbn-mathbbq-and-mathbbr, 2012.

[19] M. P. Stannett. The Stone-Čech compactification of the rational world. *Glasgow Mathematics Journal*, 1988.

[20] Norman Steenrod. *Topology of fiber bundles*. Princeton University Press, 1951.

[21] Marshall Stone. Applications of the theory of Boolean rings to general topology. *Transactions of the American Mathematical Society*, 1937.

[22] Richard G. Swan. Vector bundles and projective modules. *Transactions of the American Mathematical Society*, 1962.

[23] Ruye Wang. The schur decomposition and qr algorithm. `http://fourier.eng.hmc.edu/e176/lectures/ch1/node5.html`, 2018.

[24] Stephen Willard. *General topology*. New York: Dover Publications, 2004.